PROGRESS IN THE SOLUTION OF THE PROBLEM OF A THREE-DIMENSIONAL BODY OSCILLATING IN THE PRESENCE OF A FREE SURFACE

FINAL TECHNICAL REPORT

BY
JOHN L. HESS

Report No. DAC 67647

15 March 1969

THIS RESEARCH WAS CARRIED OUT UNDER THE NAVAL SHIP SYSTEMS COMMAND GENERAL HYDROMECHANICS RESEARCH PROGRAM SR 009 01 01, ADMINISTERED BY THE NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER.

Contract No. NONR 5054 (00)

E R R A T A

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1.0 ABSTRACT

This report summarizes results obtained under contract Nonr 5054(00). The objective of this work was the construction of a computer program for the solution of the problem of an arbitrary three-dimensional body with zero translational velocity performing small steady harmonic oscillations in the presence of an otherwise undisturbed free surface, which bounds the fluid in which the body is immersed. Because the sponsorship was terminated, this work was not carried to completion. The results that were obtained are contained in this report.

The solution of the above problem consists of two principal tasks: the development of certain analytic formulas and the incorporation of these formulas into a working computer program. Almost all of the necessary formulas were derived and tested numerically, but they could not be incorporated into a program in the time available. The subsequent sections of this report describe the general scheme for solving the problem of interest, present the formulas that were derived to implement this scheme, and point out areas where additional formulas need to be developed.

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5.0 PRINCIPAL NOTATION

A	equals νR, the dimensionless horizontal distance between a point source and a field point where the potential is to be evaluated
A_o	equals νR_o, the dimensionless horizontal distance between a field point and the centroid of an element
A_{ij}	normal velocity at the control point of the i-th surface element due to a unit source density on the j-th element
a,b,c	coordinates of the location of a point source
a_{mn}	$m = 1,2,3$, $n = 1,2,3$; components of the unit vectors along the axes of the coordinate system of an element, equation (70)
$a_k(\rho)$	coefficient functions defined by equation (58)
B	equals νh, the dimensionless vertical distance between the image of a point source and a field point where the potential is to be evaluated
B_o	equals νh_o, the dimensionless vertical distance between a field point and the image of the centroid of an element
b_{mn}	with superscripts x, y, and z, these are coefficients in the multipole expansion
$b_k(\rho)$	coefficient functions defined by equation (63)
$C_k(B)$	coefficient functions defined by equations (48) and (52)
$c_k(\rho)$	coefficient functions defined by equation (66)
c_{mn}	with superscripts x, y, and z, these are coefficients in the multipole expansion
Ei	exponential integral, equation (40)
f	function that specifies normal velocity distribution on the body surface
$f_k(\epsilon)$	coefficient functions defined by equations (64) and (67)
$G(A,\rho)$	leading term in the expansion of section 7.5 for the potential, equation (68)

$g(A,\rho)$	function defined in terms of $G(A,\rho)$ by equation (69)
H_j	weights in Laguerre-Gauss quadrature
$H_o^{(1)}$	Hankel function of first kind of order zero
$H(A,\rho)$	leading term in the expansion of section 7.5 for the A-derivative of the potential, equation (68)
h	vertical distance between the image of a point source in the free surface and a field point where the potential is to be evaluated, figure 2
h_o	vertical distance between a field point and the image of the centroid of an element
$h(A,\rho)$	function defined in terms of $H(A,\rho)$ by equation (69)
I_o, J_o	Bessel functions of first and second kinds of order zero
I_{mn}	normalized moments of the area of a quadrilateral element about its centroid, equation (72)
i,j	subscripts denoting quantities associated with the i-th and j-th elements, respectively (other uses of these subscripts with certain variables are detailed explicitly in this section)
$\vec{i},\vec{j},\vec{k}$	unit vectors along the axes of the x,y,z-coordinate system
k	integer subscript used in several ways, especially to denote terms of expansions
M	integer denoting the order of an expansion, usually the upper limit of a summation
N	number of elements used to approximate the body surface
n	distance normal to the body surface
$\vec{n}$	unit normal vector to the body surface
P	point off the body surface where the potential is to be evaluated
p	point on the body surface where the potential is to be evaluated; with subscript i, the control point of the i-th element
q	point where a source is located, especially a point on the body surface

R	horizontal distance between a point source and a field point where the potential is to be evaluated, figure 2
R_o	horizontal distance between a field point and the centroid of an element
r	distance between a point source and a field point where the potential is to be evaluated, figure 2
r_1	distance between the image of a point source in the free surface and a field point where the potential is to be evaluated, figure 2
S	mean position of the surface of the body that is performing oscillations
t	principal use is as the maximum dimension of a quadrilateral element; in certain sections also used as time, as integration variable, and with subscript j as abscissas for Laguerre-Gauss quadrature
V	with subscripts x,y,z, denotes velocity component
$\vec{V}_{ij}$	velocity at the control point of the i-th surface element due to a unit source density on the j-th element
$\vec{v}$	vector velocity
x,y,z	Cartesian coordinates. The mean position of the free surface is the plane $y = 0$, and the flow field is the half-space $y < 0$. Used to denote coordinates of a field point where the potential is to be evaluated
x_o,y_o,z_o	coordinates of the centroid of a quadrilateral element
ΔS	area of a quadrilateral element
ϵ	$B - \rho A$
ν	σ^2/g, where g is acceleration of gravity. This is the spatial circular frequency of the motion
ξ,η,ζ	coordinate system based on a quadrilateral element
ρ	a particular value of B/A about which the potential function is expanded
σ	fluid motion is a pure harmonic in time with circular frequency equal to σ. Also used as surface source density.

Φ	integrated potential of a quadrilateral element. Also time-dependent potential
Φ_{ij}	potential at the control point of the i-th element due to a unit source density on the j-th element
φ	velocity potential after harmonic time dependence has been removed. Used in various places to denote various potentials
φ_H	potential defined by equation (27)
φ_J	potential defined by equation (30)
φ_L	potential defined by equation (26)
φ_N	potential defined by equation (29)
φ_S	potential of an oscillating point source
∇^2	Laplacian operator

6.0 THE METHOD OF SOLUTION

6.1 General Description of the Method

For the past several years there has been a continuous effort at this facility directed towards the solution of certain three-dimensional problems of fluid dynamics. The first problem studied was that of the potential flow about an arbitrary three-dimensional body in an infinite fluid. This work was successfully completed, and a description of the method of solution is contained in references [1] and [2]. A subsequent investigation concerned the solution of the Helmholtz equation governing the scattering and radiation of acoustic waves by an aribtrary three-dimensional body. A description of the method of solution is contained in reference [3].

The two above-mentioned problems are logically similar, and the same general approach was adopted to effect each of their solutions. This same approach can be used to solve the problem of interest here. In each case it is desired to solve for a scalar velocity potential that satisfies a particular partial differential equation together with certain auxiliary conditions and whose normal derivative is specified on the body surface. The solution method is based on a point source function that identically satisfies the partial differential equation and the auxiliary conditions. A distribution of the appropriate type of source density is assumed to lie on the body surface. Applying the normal derivative condition on the body surface then leads to a Fredholm integral equation of the second kind for the unknown source density distribution. The only difference between the three above-mentioned problems lies in the nature of the three different point-source functions that are appropriate to the various problems.

The approximate solution of the integral equation is effected in the following manner. The body surface is approximated by a large number of small plane surface elements, over each of which the source density is assumed constant. On each element a control point is selected at which the boundary condition is to be satisfied. The basic formulas for the point source potential and its gradient are integrated over each element to obtain the effects of the elements at each others' control points per unit value of source density. In particular, a matrix is obtained whose entries are the normal derivatives of

potential induced by the elements at each others' control points for a unit value of source density. This is the coefficient matrix for a set of linear algebraic equations for the unknown values of the source density on the elements. The right-hand sides of these equations are the prescribed values of the normal derivative of the potential at the control points. Once this set of equations has been solved for the values of the source density on the elements, all other quantities of interest can be calculated fairly easily at any desired point. The key operation in this solution scheme is the integration of the basic formulas for the basic point source potential and its gradient over a plane surface element. This is also the only operation that is different for the three above-mentioned problems. Thus the solution of the problem of oscillatory motion near a free surface is primarily a matter of performing this integration numerically.

6.2 Mathematical Statement of the Problem

The problem of a body with zero translational velocity performing small steady oscillations in the presence of a free surface is well known. The mathematical formulation is given in references [4] and [5], and results will simply be stated here with no attempt at a derivation.

Cartesian coordinates x,y,z are assumed, and the undisturbed location of the free surface is taken as the plane $y = 0$. The fluid motion takes place in the half space $y < 0$, (figure 1). In discussing this problem distance perpendicular to the free surface, i.e., y distance, is denoted vertical distance and distance parallel to the free surface, i.e., x and z distance, is denoted horizontal distance. The surface of the body that is performing small oscillations is denoted S. Specifically, S represents the mean position of the body surface, and the oscillations take place about this surface. In particular, the surface S is independent of time. The body represented by S may be surface-piercing, as shown in figure 1, or completely submerged. Moreover, S may be multiply-connected and represent an ensemble of bodies.

The fluid velocity field in the half-space $y < 0$ is assumed to be irrotational. Thus, the velocity is equal to the negative gradient of a scalar potential function $\Phi(x,y,z,t)$, which is a function of the time t as well

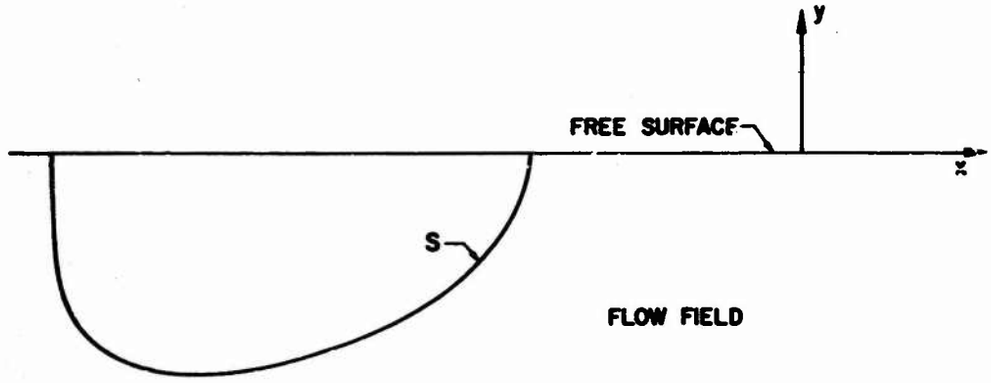

Figure 1. - A three-dimensional body near a free-surface.

as of position. By assumption the fluid motion is harmonic with a single circular frequency σ. Thus, the time-dependent potential $\Phi(x,y,z,t)$ can be written

$$\Phi(x,y,z,t) = \mathrm{Re}[\varphi(x,y,z)e^{-i\sigma t}] \tag{1}$$

where the potential $\varphi(x,y,z)$ is independent of time and is thus denoted the steady potential. It is the potential φ that must be calculated.

According to references [4] and [5] φ must satisfy the following equations:

$$\nabla^2\varphi = 0 \qquad \text{for} \qquad y < 0 \tag{2}$$

$$\frac{\partial\varphi}{\partial y} - \nu\varphi = 0 \qquad \text{for} \qquad y = 0 \tag{3}$$

$$\lim_{y \to -\infty} (\text{grad } \varphi) = 0 \tag{4}$$

$$\lim_{\rho \to \infty} \left[\sqrt{\rho} \left(i \frac{\partial \varphi}{\partial \rho} + \nu \varphi \right) \right] = 0, \qquad \rho^2 = x^2 + z^2 \tag{5}$$

$$\frac{\partial \varphi}{\partial n} = f(S) \quad \text{on} \quad S \tag{6}$$

The parameter ν is defined as

$$\nu = \frac{\sigma^2}{g} \tag{7}$$

where g is the acceleration of gravity. In (6) n denotes distance normal to S.

The physical significance of the above equations is as follows. Equation (2) is the partial differential equation for φ and expresses the conditions of incompressibility and irrotationality. Equations (3), (4), and (5) are the auxiliary conditions on φ. Equation (3) is the linearized free-surface condition, which requires that the pressure on the free-surface be constant. Equation (4) expresses the vanishing of the disturbance at infinite depth, and equation (5) is the radiation condition that requires the disturbance to be an outgoing wave at infinite horizontal distance. Equation (6) is the boundary condition on S. It expresses the fact that the normal fluid velocity on S must be specified as a function $f(S)$ of position on the surface. Often the fluid normal velocity is specified as equal to the normal velocity of the surface S. However, in the case of a known incident wave, φ denotes the disturbance potential due to the body, and the boundary condition expresses the fact that the normal velocity of the disturbance must cancel that of the incident wave on the body surface.

6.3 The Oscillating Point-Source Potential

The method of solving the mathematical problem defined by equations (2), (3), (4), (5), and (6) is based on a so-called elementary solution or point-source solution. The point-source potential is defined as the one that identically satisfies the auxiliary conditions (3), (4), and (5), and that satisfies the homogeneous partial differential equation (2) throughout space except at one point, where it is singular. The singularity is such that the point-source potential satisfies equation (2) with the right side replaced by

the Dirac delta function (a function zero everywhere except at one point where it is infinite and such that the integral of the function over any volume that includes the singularity is unity). The point where the point-source potential is singular is the location of the point source.

Expressions for the point-source potential are given in reference [4]. If the source is located at the point (a,b,c) its potential at a general point (x,y,z) is

$$\varphi_s = \frac{1}{r} + \frac{1}{r_1} + 2\nu e^{\nu y} \int_{\infty}^{y} \frac{e^{-\nu y}}{r_1}\, dy + 2\pi i \nu e^{\nu(y+b)}\, H_o^{(1)}(\nu R) \tag{8}$$

where

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2 \tag{9}$$

$$r_1^2 = (x-a)^2 + (y+b)^2 + (z-c)^2 \tag{10}$$

$$R^2 = (x-a)^2 + (z-c)^2 \tag{11}$$

and where $H_o^{(1)} = J_o + iY_o$ is the Hankel function of the first kind of order zero. (J_o and Y_o are the ordinary Bessel functions of order zero.) The physical significances of the three distances r, r_1, and R are evident from equations (9), (10), and (11), and they are illustrated in figure 2. The distance r is the distance between the point source and the field point where the potential is evaluated. Similarly, r_1 is the distance from the field point to the image point of the source in the free surface. Finally, R is the horizontal distance between source and field point. The point source whose potential is given by (8) is defined as a source of unit strength.

For ordinary problems governed by Laplace's equation with no free surface the point-source potential is simply $1/r$. The first two terms of (8) are seen to be the potentials of $1/r$-type point sources – one located at the point source and one at its image in the free surface. The integral term in (8) may be written

$$2\nu e^{\nu y} \int_{\infty}^{y} \frac{e^{-\nu y}}{r_1}\, dy = -2\nu \int_{0}^{\infty} \frac{e^{-\nu u}\, du}{\sqrt{R^2 + (u+y+b)^2}} \tag{12}$$

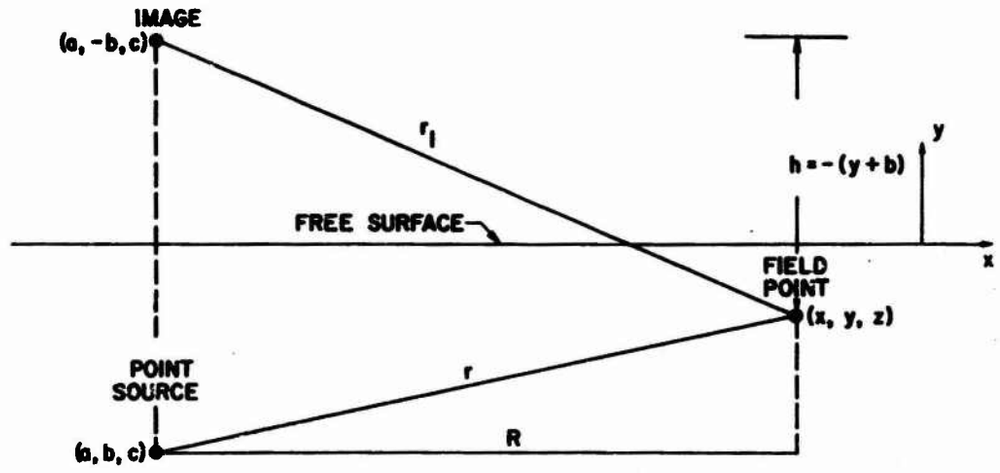

Figure 2. - Distances between a field point where potential is evaluated and a point source and its image.

The square root on the right side of (12) is the distance between the point (x,y,z) and the point (a,-b-u, c). Thus, this integral represents the potential of a 1/r-type line source of exponentially decaying strength starting at the image point (a, -b, c) of the point source and running vertically downward through the free surface to $y = -\infty$.

For large values of νR, the function $H_o^{(1)}(\nu R)$ oscillates with increasing horizontal distance R at a circular frequency of ν. Thus ν is denoted the spatial circular frequency of the motion, and its relation to the temporal frequency σ is given by (7).

6.4 Representation of the Solution by a Source Distribution on the Body Surface

The unit point-source potential, which for the present problem is given by (8), is a function of the field point P with coordinates x,y,z where the potential is evaluated and of the point q with coordinates a,b,c where the source is located. Accordingly, the potential of a unit point source may be written $\varphi_s(P,q)$. Now let the source point q be restricted to lie on the surface S. (Figure 3.) A distribution of source density of strength $\sigma(q)$

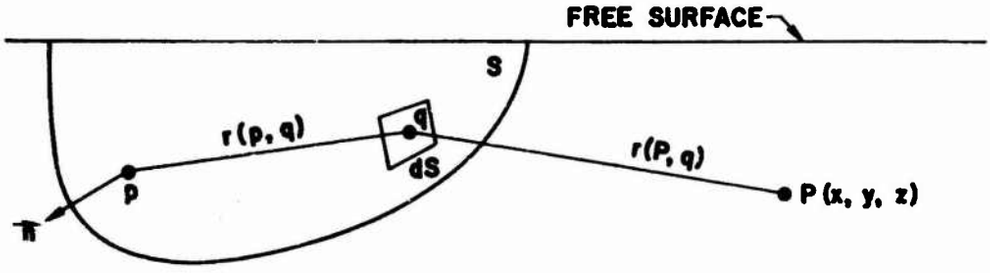

Figure 3. - Illustration of quantities appearing in the integral equation.

is assumed to lie on the surface S. The potential at P due to this distribution is

$$\varphi(P) = \iint_S \varphi_s(P,q)\,\sigma(q)\,dS \tag{13}$$

This potential satisfies the auxiliary conditions (3), (4), and (5) simply because φ_s does and because all these conditions are linear. Similarly $\varphi(P)$ satisfies the partial differential equation (2) at all points of the half space $y < 0$ that are exterior to the surface S. Thus $\varphi(P)$ satisfies equations (2), (3), (4), and (5) regardless of the nature of the function $\sigma(q)$. This function is determined in such a way that $\varphi(P)$ satisfies the boundary condition (6).

Applying the boundary condition (6) to the potential of (13) requires evaluating the limits of the spatial derivatives of (13) as the point P approaches a point p on the surface S (figure 3). These same limits are required to calculate fluid velocity components on S. Care is required because the derivatives of $1/r$, where $r = r(P,q)$ is given by (9), become singular as the surface is approached. The other terms that comprise φ_s in

equation (8) are not singular as $P \to p$. Thus the singularity is exactly the same as that encountered in references [1] and [2], and the results of these references apply here. In particular, the application of the boundary condition (6) to the potential (13) gives

$$-2\pi\sigma(p) + \iint_S \frac{\partial}{\partial n}[\varphi_s(p,q)]\sigma(q)dS = f(p) \tag{14}$$

where the right side of (6) has been written $f(p)$ to show its dependence on position on S. Equation (14) is a Fredholm integral equation of the second kind for the source distribution $\sigma(q)$. Once it is solved for $\sigma(q)$ the potential at any point is given by (13) and the velocity at any point by

$$\vec{v}(P) = -\text{ grad } \varphi(P) = -\iint_S \text{grad } [\varphi_s(P,q)]\sigma(q)\, dS \tag{15}$$

The evaluation of either (13) or (15) at a point $P = p$ on S requires the limiting process discussed in references [1] and [2].

6.5 Logic of the Numerical Procedure

It should be noted that the discussion of the previous section applies to the problems of references [1], [2], [3] as well as to the present problem. In fact the equations of section 6.4 apply exactly to each of the problems of these references if φ_s is set equal to the appropriate point-source function. Similarly the method of numerically solving the problem defined by equation (14) and subsequently calculating the quantities (13) and (15) is logically identical for all problems. This method is described in great detail in the references. It will simply be outlined here.

The body surface S is approximated by a large number of small plane quadrilateral surface elements, as is shown schematically in figure 4. The total number of elements used to approximate the surface is denoted N, and typical elements are denoted the i-th and the j-th, as shown in figure 4. Subscripts i and j denote quantities associated with the i-th and j-th elements, respectively. In particular $\vec{n}_i$ is the unit normal vector to the i-th element,

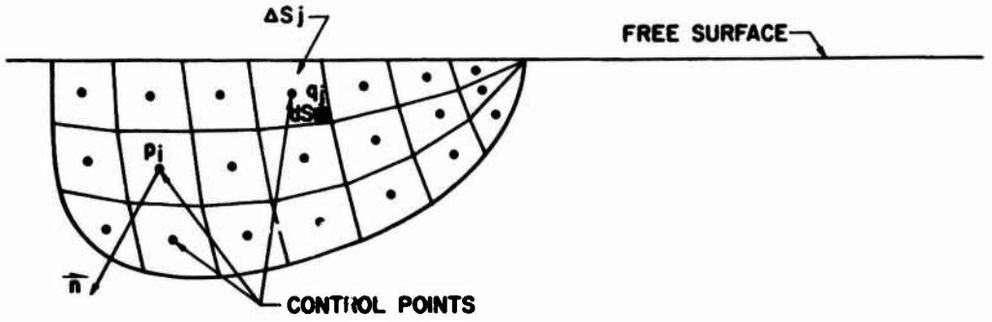

Figure 4. - Approximation of the body surface by plane quadrilateral elements.

and ΔS_j is the area of the j-th element. On each element the source density is taken as constant. The value of source density on the j-th element is denoted σ_j. On each element a control point is selected where the integral equation (14) is required to be satisfied and where the potential and velocity will eventually be evaluated. The control point of the i-th element is denoted p_i.

It is clear from the above that the potential and velocity at the control point of the i-th element due to the j-th element are, respectively,

$$\Phi_{ij}\sigma_j \qquad \text{and} \qquad \vec{V}_{ij}\sigma_j \tag{16}$$

where

$$\Phi_{ij} = \iint_{\Delta S_j} \varphi_s(p_i, q_j)\, ds$$

$$\vec{V}_{ij} = -\iint_{\Delta S_j} \operatorname{grad}\,[\varphi_s(p_i, q_j)]\, dS \tag{17}$$

and where q_j denotes a general point of the j-th element as shown in figure 4. The normal velocity at the control point of the i-th element due to all the elements is

$$V_{ni} = \sum_{j=1}^{N} A_{ij}\sigma_j \tag{18}$$

where

$$A_{ij} = \vec{n}_i \cdot \vec{V}_{ij} \tag{19}$$

Application of the normal derivative boundary condition at the control points of all elements then gives

$$\sum_{j=1}^{N} A_{ij}\sigma_j = -f(p_i), \qquad i = 1, 2, \ldots N \tag{20}$$

where $f(p_i)$ is the right side of (6) or (14) evaluated at p_i. (Recall that $\partial\varphi/\partial n = -V_n$.) Equation (20) is the numerical approximation to the integral equation (14). It consists of a set of linear algebraic equations for the values of source density on the elements. Once this set of equations is solved, the potentials and velocities at the control points of the elements are calculated from

$$\begin{aligned} \varphi_i &= \sum_{j=1}^{N} \Phi_{ij}\sigma_j, \qquad i = 1, 2, \ldots N \\ \vec{V}_i &= \sum_{j=1}^{N} \vec{V}_{ij}\sigma_j, \qquad i = 1, 2, \ldots N \end{aligned} \tag{21}$$

All of the above is quite straightforward. The only problem is the calculation of the integrals in (17) for a function φ_s given by (8). That is, the only new calculation that is required to solve the problem of oscillatory motion near a free surface is the integration of the oscillating point-source potential and its derivatives over a quadrilateral element.

7.0 CALCULATION OF THE OSCILLATING POINT-SOURCE POTENTIAL AND ITS DERIVATIVES

7.1 General Remarks

As stated in the previous section, the central problem of this method of solution is the integration of the oscillating point-source potential and its derivatives over a plane quadrilateral surface element. These integrations must be done numerically and they must be done very rapidly, because a total of N^2 such integrations are required to produce the matrices Φ_{ij} and $\vec{V}_{ij}$, where N is the number of elements used to approximate the body surface and is thus a large number. The problem of numerical integration divides itself into two parts. The first is the calculation of the point-source potential and derivatives for general values of the parameters. The second is the integration of these quantities over an element by means of a multipole expansion. This section, 7.0, will consider the first of these parts, and section 8.0 will consider the second.

7.2 The Quantities Calculated

The horizontal distance R between the point source at the point (a,b,c) and the field point (x,y,z) is given by equation (11). The vertical distance between the field point and the image point $(a,-b,c)$ of the point source is

$$h = -(y + b) > 0 \tag{22}$$

(See figure 2 for an illustration of these quantities.) Using these variables and the results of equation (12) the oscillating point-source potential (8) can be written

$$\varphi_s = \frac{1}{r} + \frac{1}{r_1} - 2\nu \int_0^\infty \frac{e^{-\nu u}\,du}{\sqrt{R^2 + (u - h)^2}} + 2\pi i e^{-\nu h} H_o^{(1)}(\nu R) \tag{23}$$

The $1/r$ and $1/r_1$ terms are already accounted for by the method of reference [1]. They will not be considered here, but attention is directed to the last two terms. It should be noticed that x and z do not enter the

above equation separately but occur only in R, i.e., the potential is axisymmetric about a vertical line through the point source. Thus, only h- and R-derivatives need be considered, and the x- and z-derivatives are easily obtained from the latter. All distances in the last two terms of (23) enter as products with the inverse distance ν. Such products are dimensionless and are now defined as new variables. In particular the dimensionless horizontal and vertical distances are

$$\begin{aligned} A &= \nu R \\ B &= \nu h \end{aligned} \tag{24}$$

These are essentially distances measured in wave lengths of the motion. Substituting these in above, together with $t = \nu u$ gives the following form for the oscillating point-source potential:

$$\varphi_s = \frac{1}{r} + \frac{1}{r_1} + \nu[-\varphi_L + \varphi_H] \tag{25}$$

where

$$\varphi_L = \varphi_L(A,B) = 2\int_0^\infty \frac{e^{-t}\,dt}{\sqrt{A^2 + (t-B)^2}} \tag{26}$$

$$\varphi_H = \varphi_H(A,B) = 2\pi i e^{-B} H_o^{(1)}(A) \tag{27}$$

This form of φ_s is used for $A > 5$. For $A \leq 5$ the terms are regrouped in the following manner:

$$\varphi_s = \frac{1}{r} + \frac{1}{r_1} + \nu[-\varphi_N + \varphi_J] \tag{28}$$

where

$$\varphi_N = \varphi_N(A,B) = 2\int_0^\infty \frac{e^{-t}\,dt}{\sqrt{A^2 + (t-B)^2}} + 2\pi e^{-B} Y_o(A) \tag{29}$$

$$\varphi_J = \varphi_J(A,B) = 2\pi i e^{-B} J_o(A) \tag{30}$$

The value $A = 5$, where the change is made from (25) to (28), is more or less arbitrary. Form (25) is more convenient at large values of A. Form (28) must be used at small values of A, because φ_N and φ_J are well-behaved at $A = 0$, while φ_L and φ_H are singular there.

It is the potentials φ_L, φ_H, φ_N, φ_J and their derivatives that are considered in this report. Only the A-derivatives need be considered. The B-derivatives can be expressed in terms of the potential itself. Specifically,

$$\frac{\partial \varphi}{\partial B} = -\varphi \qquad \text{if} \qquad \varphi = \varphi_H \quad \text{or} \quad \varphi_J \tag{31}$$

Also

$$\frac{\partial \varphi}{\partial B} = -\varphi + \frac{2}{\sqrt{A^2 + B^2}}$$

$$= -\varphi + \frac{2}{vr_1}, \qquad \text{if} \qquad \varphi = \varphi_L \quad \text{or} \quad \varphi_N \tag{32}$$

The additional term in (32) can be handled by the method of reference [1].

For the integration over an element discussed in section 8.0 all four of the above potentials must be considered. In the present section, however, the interest is in formulas for expressing the point-source potential itself. The two potentials φ_B and φ_J each consist of an exponential times a Bessel function, and these functions may be calculated from more or less standard formulas. (Reference [6] contains a rather extensive section on Bessel functions.)

Thus, in this section attention is concentrated on the potentials φ_L and φ_N and on their A-derivatives. Generally φ_L is of interest for $A > 5$ and φ_N for $A \leq 5$, but both have been investigated over a wider range of A so that the cross-over point could be changed if desired. The potentials φ_L and φ_N and their A-derivatives are shown in figures 7, 8, 9, and 10, and numerical values are given in tables 1 and 2. The nature and general magnitude of these functions are illustrated in these figures.

TABLE 1

THE POTENTIAL φ_L AND ITS DERIVATIVE $\partial\varphi_L/\partial A$

	A = 0.5		A = 1.0		A = 2.0		A = 3.0		A = 4.0		A = 6.0		A = 10.0	
B	φ_L	$\partial\varphi_L/\partial A$	φ_L	$\partial\varphi_L/\partial A$	φ_L	$\partial\varphi_L/\partial A$	φ_L	$\partial\varphi_L/\partial A$	φ_L	$\partial\varphi_L/\partial A$	φ_L	$\partial\varphi_L/\partial A$	φ_L	$\partial\varphi_L/\partial A$
0	2.368995	-2.786669	1.509218	-1.077723	0.881163	-0.368120	0.620326	-0.184774	0.477381	-0.110525	0.325592	-0.052043	0.198148	-0.019469
0.25	2.694506	-3.741952	1.613615	-1.267702	0.906841	-0.396384	0.630396	-0.192887	0.482308	-0.113669	0.327282	-0.052809	0.198553	-0.019585
0.5	2.803120	-3.820641	1.669149	-1.347642	0.923434	-0.413405	0.637209	-0.198169	0.485708	-0.115793	0.328468	-0.053340	0.198840	-0.019667
0.75	2.734840	-3.411067	1.674101	-1.318037	0.930054	-0.417826	0.640540	-0.200377	0.487505	-0.116820	0.329135	-0.053627	0.199008	-0.019714
1.0	2.568328	-2.874357	1.636019	-1.214566	0.926729	-0.410180	0.640326	-0.199461	0.487670	-0.116727	0.329278	-0.053665	0.199055	-0.019726
1.25	2.359106	-2.3576	1.56751	-1.075428	0.914288	-0.392453	0.636662	-0.195611	0.486218	-0.115541	0.328896	-0.053455	0.198983	-0.019703
1.5	2.139242	-1.90695	1.48056	-0.927444	0.894102	-0.367354	0.629786	-0.189194	0.483207	-0.113335	0.327997	-0.053003	0.198792	-0.019645
1.75	1.925896	-1.53018	1.384584	-0.785896	0.867797	-0.337663	0.620040	-0.180692	0.478734	-0.110221	0.326594	-0.052321	0.198481	-0.019552
2.0	1.727562	-1.221976	1.286213	-0.658092	0.837027	-0.305808	0.607837	-0.170635	0.472926	-0.106334	0.324707	-0.051425	0.198055	-0.019427
2.25	1.54781	-0.972920	1.189827	-0.546619	0.803310	-0.273684	0.593621	-0.159546	0.465930	-0.101827	0.322363	-0.050333	0.197513	-0.019268
2.5	1.38748	-0.773163	1.098099	-0.451514	0.767947	-0.242637	0.577839	-0.147898	0.457909	-0.096852	0.319591	-0.049068	0.196860	-0.019079
2.75	1.245927	-0.613715	1.012504	-0.371561	0.731994	-0.213530	0.560911	-0.136093	0.449030	-0.091561	0.316425	-0.047655	0.196098	-0.018860
3.0	1.121754	-0.486847	0.933709	-0.305028	0.696270	-0.186850	0.543221	-0.124452	0.439459	-0.086088	0.312902	-0.046117	0.195232	-0.018614
3.25	1.013255	-0.386123	0.861858	-0.250056	0.661379	-0.162807	0.525105	-0.113218	0.429352	-0.080554	0.309058	-0.044481	0.194266	-0.018341
3.5	0.918650	-0.306272	0.796764	-0.204867	0.627745	-0.141418	0.506848	-0.102561	0.418858	-0.075060	0.304934	-0.042769	0.193204	-0.018045
3.75	0.836217	-0.243030	0.738049	-0.167853	0.595648	-0.122580	0.488681	-0.092591	0.408109	-0.069688	0.300567	-0.041006	0.192051	-0.017728
4.0	0.764363	-0.192973	0.685231	-0.137615	0.565254	-0.106114	0.470793	-0.083356	0.397224	-0.064501	0.295996	-0.039212	0.190813	-0.017391
4.25	0.701653	-0.153365	0.637792	-0.112947	0.536642	-0.091808	0.453326	-0.074910	0.386305	-0.059547	0.291256	-0.037407	0.189494	-0.017037
4.5	0.646823	-0.122026	0.595202	-0.092848	0.509828	-0.079434	0.436387	-0.067215	0.375459	-0.054857	0.286383	-0.035608	0.188101	-0.016669
4.75	0.598683	-0.097227	0.556957	-0.076478	0.484781	-0.068766	0.420053	-0.060255	0.364698	-0.050451	0.281408	-0.033830	0.186638	-0.016287
5.0	0.556515	-0.077597	0.522582	-0.063144	0.461441	-0.059592	0.404371	-0.053991	0.354141	-0.046337	0.276362	-0.032085	0.185111	-0.015896
5.25	0.519258	-0.062049	0.491644	-0.052278	0.439726	-0.051714	0.389372	-0.048375	0.343815	-0.042518	0.271272	-0.030384	0.183527	-0.015496
5.5	0.486287	-0.049728	0.463748	-0.043417	0.419543	-0.044958	0.375066	-0.043355	0.333755	-0.038988	0.266163	-0.028735	0.181889	-0.015089
5.75	0.457001	-0.039954	0.438544	-0.036181	0.400792	-0.039164	0.361454	-0.038880	0.323988	-0.035737	0.261057	-0.027145	0.180205	-0.014679
6.0	0.430889	-0.032193	0.415719	-0.030264	0.383373	-0.034197	0.348523	-0.034896	0.314532	-0.032754	0.255972	-0.025618	0.178479	-0.014266
6.25	0.407514	-0.026024	0.394997	-0.025418	0.367187	-0.029935	0.336256	-0.031354	0.305400	-0.030022	0.250927	-0.024158	0.176716	-0.013852
6.5	0.386508	-0.021112	0.376135	-0.021439	0.352136	-0.026275	0.324630	-0.028207	0.296597	-0.027527	0.245935	-0.022766	0.174922	-0.013438
6.75	0.367556	-0.017195	0.358920	-0.018165	0.338132	-0.023128	0.313618	-0.025413	0.288125	-0.025251	0.241010	-0.021443	0.173101	-0.013027
7.0	0.350391	-0.014066	0.343165	-0.015464	0.325087	-0.020419	0.303191	-0.022932	0.279983	-0.023177	0.236161	-0.020190	0.171258	-0.012619
7.25	0.334785	-0.011560	0.328708	-0.013230	0.312923	-0.018082	0.293319	-0.020728	0.272166	-0.021290	0.231397	-0.019004	0.169397	-0.012216
7.5	0.320543	-0.009550	0.315404	-0.011375	0.301566	-0.016063	0.283972	-0.018769	0.264667	-0.019573	0.226725	-0.017886	0.167522	-0.011818
7.75	0.307500	-0.007933	0.303130	-0.009830	0.290950	-0.014313	0.275120	-0.017026	0.257477	-0.018012	0.222152	-0.016832	0.165637	-0.011427
8.0	0.295514	-0.006628	0.291776	-0.008539	0.281011	-0.012794	0.266733	-0.015475	0.250586	-0.016592	0.217680	-0.015842	0.163746	-0.011043
8.25	0.284462	-0.005571	0.281247	-0.007455	0.271694	-0.011472	0.258785	-0.014092	0.243983	-0.015302	0.213310	-0.014911	0.161852	-0.010667
8.5	0.274240	-0.004713	0.271458	-0.006542	0.262947	-0.010319	0.251246	-0.012857	0.237658	-0.014128	0.209052	-0.014037	0.159958	-0.010300
8.75	0.264758	-0.004013	0.262336	-0.005776	0.254725	-0.009309	0.244093	-0.011754	0.231599	-0.013059	0.204900	-0.013218	0.158068	-0.009941
9.0	0.255937	-0.003440	0.253816	-0.005114	0.246984	-0.008424	0.237300	-0.010766	0.225795	-0.012086	0.200857	-0.012451	0.156183	-0.009592
9.25	0.247708	-0.002968	0.245841	-0.004554	0.239686	-0.007644	0.230845	-0.009880	0.220233	-0.011200	0.196922	-0.011733	0.154306	-0.009252
9.5	0.240014	-0.002579	0.238361	-0.004074	0.232796	-0.006957	0.224706	-0.009085	0.214902	-0.010392	0.193094	-0.011061	0.152440	-0.008922
9.75	0.232803	-0.002255	0.231331	-0.003660	0.226283	-0.006348	0.218862	-0.008369	0.209792	-0.009654	0.189373	-0.010431	0.150587	-0.008602
10.0	0.226028	-0.001985	0.224712	-0.003303	0.220117	-0.005808	0.213297	-0.007723	0.204892	-0.008980	0.185756	-0.009843	0.148747	-0.008292

TABLE 2

THE POTENTIAL φ_N AND ITS DERIVATIVE $\partial\varphi_N/\partial A$

	A = 0.0	A = 0.5		A = 1.0		A = 2.0		A = 3.0		A = 4.0		A = 6.0		A = 10.0	
B	φ_N	φ_N	$\partial\varphi_N/\partial A$	φ_N	$\partial\varphi_N/\partial A$	φ_N	$\partial\varphi_N/\partial A$	φ_N	$\partial\varphi_N/\partial A$	φ_N	$\partial\varphi_N/\partial A$	φ_N	$\partial\varphi_N/\partial A$	φ_N	$\partial\varphi_N/\partial A$
0	-∞	-0.423999	6.458865	2.063753	3.830782	4.087948	0.304385	2.988144	-2.224764	0.370939	-2.610765	-1.485189	1.047579	0.457940	-1.584323
0.25	-0.845066	0.519320	3.458477	2.044887	2.555045	3.404288	0.127363	2.474455	-1.781622	0.399411	-2.060859	-1.082955	0.803578	0.470971	-1.238294
0.5	0.550997	1.109084	1.787059	2.005491	1.629517	2.868447	-0.005510	2.073363	-1.435485	0.421148	-1.632266	-0.769826	0.613615	0.421000	-0.968799
0.75	1.140607	1.415523	0.956214	1.936045	1.000577	2.444832	-0.100157	1.759018	-1.164000	0.437225	-1.297850	-0.526217	0.465798	0.364238	-0.758899
1.0	1.394350	1.540843	0.526885	1.840021	0.591172	2.106439	-0.162779	1.511398	-0.949931	0.448512	-1.036514	-0.336871	0.350863	0.327736	-0.595403
1.25	1.478965	1.558900	0.291290	1.726387	0.330882	1.833047	-0.199777	1.315053	-0.780078	0.455722	-0.831872	-0.189901	0.261592	0.299200	-0.468041
1.5	1.473233	1.516041	0.156007	1.604293	0.167791	1.609632	-0.217298	1.158118	-0.644377	0.459457	-0.671214	-0.076043	0.192356	0.276841	-0.368811
1.75	1.419265	1.440547	0.076453	1.480948	0.067704	1.425053	-0.220799	1.031505	-0.535189	0.460237	-0.544698	0.011928	0.138765	0.259266	-0.291483
2.0	1.340965	1.349571	0.029271	1.361261	0.006202	1.271018	-0.214794	0.928286	-0.446718	0.458521	-0.444705	0.079645	0.097393	0.245394	-0.231207
2.25	1.252263	1.253431	0.001552	1.248275	-0.029266	1.141303	-0.202803	0.843187	-0.374559	0.454711	-0.365350	0.131508	0.065566	0.234381	-0.184202
2.5	1.161300	1.158217	-0.014244	1.143618	-0.048599	1.031176	-0.187437	0.772201	-0.315351	0.449172	-0.302084	0.170953	0.041194	0.225573	-0.147530
2.75	1.072747	1.067377	-0.022668	1.047954	-0.057771	0.936997	-0.170538	0.712281	-0.266505	0.442225	-0.251396	0.200666	0.022641	0.218459	-0.118898
3.0	0.989153	0.982699	-0.026539	0.961318	-0.060648	0.855926	-0.153368	0.661108	-0.226017	0.434160	-0.210568	0.222749	0.008630	0.212647	-0.096523
3.25	0.911760	0.904959	-0.027035	0.883360	-0.059733	0.785720	-0.136731	0.616915	-0.192317	0.425225	-0.177499	0.238846	-0.001844	0.207829	-0.079017
3.5	0.841018	0.834309	-0.027081	0.813509	-0.056643	0.724582	-0.121110	0.578350	-0.164163	0.415644	-0.150561	0.250253	-0.009563	0.203767	-0.065299
3.75	0.776899	0.770532	-0.025596	0.751090	-0.052416	0.671004	-0.106764	0.544367	-0.140567	0.405606	-0.128488	0.257982	-0.015145	0.200277	-0.054530
4.0	0.719104	0.713208	-0.023635	0.695388	-0.047711	0.623968	-0.093797	0.514161	-0.120730	0.395274	-0.110235	0.262830	-0.019072	0.197220	-0.046052
4.25	0.667183	0.661813	-0.021485	0.645702	-0.042931	0.582384	-0.082215	0.487101	-0.104009	0.384787	-0.095211	0.265427	-0.021722	0.194484	-0.039358
4.5	0.620630	0.615796	-0.019317	0.601362	-0.038319	0.545452	-0.071963	0.462691	-0.089877	0.374257	-0.082632	0.266277	-0.023392	0.191987	-0.034053
4.75	0.578915	0.574599	-0.017238	0.561755	-0.034011	0.512525	-0.062948	0.440539	-0.077904	0.363777	-0.072082	0.265742	-0.024316	0.189664	-0.029826
5.0	0.541533	0.537696	-0.015301	0.526318	-0.030071	0.483048	-0.055061	0.420325	-0.067736	0.353424	-0.063183	0.264161	-0.024676	0.187468	-0.026440
5.25	0.507996	0.504602	-0.013533	0.494554	-0.026521	0.456554	-0.048185	0.401797	-0.059080	0.343256	-0.055638	0.261770	-0.024614	0.185363	-0.023708
5.5	0.477869	0.474873	-0.011944	0.466014	-0.023357	0.432648	-0.042210	0.384743	-0.051692	0.333320	-0.051706	0.258763	-0.024241	0.183319	-0.021484
5.75	0.450753	0.448112	-0.010528	0.440309	-0.020558	0.410998	-0.037024	0.368990	-0.045373	0.323649	-0.043695	0.255294	-0.023645	0.181318	-0.019660
6.0	0.426294	0.423966	-0.009276	0.417094	-0.018097	0.391322	-0.032550	0.354392	-0.039953	0.314268	-0.038951	0.251484	-0.022892	0.179346	-0.018145
6.25	0.404171	0.402122	-0.008176	0.396008	-0.015942	0.373378	-0.028637	0.340827	-0.035292	0.305195	-0.034849	0.247431	-0.022035	0.177391	-0.016873
6.5	0.385097	0.382309	-0.007212	0.376969	-0.014057	0.356957	-0.025264	0.328190	-0.031274	0.296437	-0.031286	0.243213	-0.021113	0.175448	-0.015791
6.75	0.365880	0.364286	-0.006370	0.359569	-0.012418	0.341887	-0.022341	0.316390	-0.027802	0.288000	-0.028178	0.238890	-0.020155	0.173511	-0.014859
7.0	0.349259	0.347844	-0.005635	0.343671	-0.010988	0.328011	-0.019806	0.305350	-0.024792	0.279886	-0.025457	0.234510	-0.019187	0.171577	-0.014046
7.25	0.334049	0.332802	-0.004994	0.329102	-0.009744	0.315200	-0.017604	0.295001	-0.022177	0.272090	-0.023066	0.230111	-0.018223	0.169645	-0.013327
7.5	0.320104	0.318998	-0.004437	0.315711	-0.008660	0.303340	-0.015691	0.285282	-0.019897	0.264608	-0.020956	0.225724	-0.017278	0.167715	-0.012683
7.75	0.307266	0.306297	-0.003956	0.303369	-0.007710	0.292331	-0.014023	0.276140	-0.017905	0.257431	-0.019090	0.221372	-0.016358	0.165788	-0.012101
8.0	0.295400	0.294577	-0.003526	0.291962	-0.006892	0.282087	-0.012568	0.267527	-0.016159	0.250550	-0.017431	0.217073	-0.015473	0.163863	-0.011568
8.25	0.284504	0.283752	-0.003150	0.281392	-0.006173	0.272523	-0.011296	0.259404	-0.014625	0.243955	-0.015955	0.212839	-0.014624	0.161943	-0.011076
8.5	0.274331	0.273672	-0.002832	0.271571	-0.005544	0.263599	-0.010182	0.251728	-0.013272	0.237636	-0.014637	0.208684	-0.013813	0.160039	-0.010618
8.75	0.264902	0.264315	-0.002548	0.262434	-0.004992	0.255233	-0.009203	0.244468	-0.012077	0.231582	-0.013455	0.204613	-0.013044	0.158163	-0.010189
9.0	0.256169	0.255592	-0.002289	0.253884	-0.004508	0.247380	-0.008341	0.237592	-0.011018	0.225782	-0.012395	0.200634	-0.012319	0.156326	-0.009785
9.25	0.247880	0.247440	-0.002	0.245894	-0.004082	0.239994	-0.007580	0.231073	-0.010076	0.220223	-0.011440	0.196748	-0.011627	0.154540	-0.009402
9.5	0.240241	0.239805	-0.001847	0.238403	-0.003706	0.233036	-0.006906	0.224883	-0.009237	0.214894	-0.010579	0.192958	-0.010973	0.152806	-0.009039
9.75	0.232932	0.232640	-0.001716	0.231303	-0.003374	0.226470	-0.006309	0.219000	-0.008488	0.209786	-0.009800	0.189267	-0.010367	0.151067	-0.008695
10.0	0.226294	0.225901	-0.001505	0.224737	-0.003080	0.220263	-0.005777	0.213404	-0.007810	0.204887	-0.009094	0.185671	-0.009795	0.149463	-0.008363

7.3 Laguerre-Gauss Quadrature for Large Values of Horizontal Distance

For "large" values of the dimensionless horizontal distance A between the point source and the field point, the oscillating point-source potential is used in the form (25), and the potential φ_L from (26) is the one that must be evaluated. Laguerre-Gauss quadrature is used to evaluate φ_L in the form

$$\varphi_L = 2\int_0^\infty \frac{e^{-t}\,dt}{\sqrt{A + (t - B)^2}} \approx 2\sum_{j=1}^{M} \frac{H_j}{\sqrt{A^2 + (t_j - B)^2}} \tag{33}$$

The constants H_j are the weights of the quadrature, and the numbers t_j are the abscissas of the quadrature. Both H_j and t_j depend only on M, the order of the quadrature. For the present application, the first abscissa was set equal to zero, i.e.,

$$t_1 = 0 \tag{34}$$

so that

$$\begin{aligned}\varphi_L &\approx \frac{2H_1}{\sqrt{A^2 + B^2}} + 2\sum_{j=2}^{M} \frac{H_j}{\sqrt{A^2 + (t_j - B)^2}} \\ &\approx \frac{2H_1}{\nu r_1} + 2\sum_{j=2}^{M} \frac{H_j}{\sqrt{A^2 + (t_j - B)^2}}\end{aligned} \tag{35}$$

In general the accuracy of Laguerre-Gauss quadrature improves with increasing M, i.e., the more abscissas, the more accuracy. The term with abscissa of zero, i.e., the first term of (35), is computationally free, because it may be combined with the $1/r_1$-term of (25). Thus, for the same computational labor the form (35) should be more accurate than the form of Laguerre-Gauss quadrature that does not prescribe any abscissas, because the former uses one more abscissa. This plausible hypothesis was verified in the present case by a large amount of numerical experimentation.

Reference [7] gives the abscissas t_j and weights H_j for the form of Laguerre-Gauss quadrature that has zero value of the first abscissa. The results for $M = 1, 2, 3, 4, 5, 6$ are given in table 3. These were used in equation (33) or (35) and the results were compared with the results of "brute force" numerical evaluations of φ_L and its A-derivative. (The approximation to the A-derivative of φ_L is obtained by analytically differentiating (33) or (35).) In this manner computational errors were obtained for various values of M at sets of values of A and B. For each A the largest error for any value of B was determined and designated the maximum error at that A. These maximum errors are presented in table 4 for $2 \leq A \leq 10$. These errors should be compared with the functions φ_L and $\partial\varphi_L/\partial A$, which are given in table 1. Table 4 illustrates the very rapid decrease of error with increasing A and M. If an allowable error is defined as one less than 0.001, two abscissas are sufficient for $A > 6$, while three suffice for $A > 4$. Thus, in the present context A becomes "large" at a value of four or less. The smaller the value of A, the larger is the value of M that must be used to obtain a given accuracy.

For the present purpose, it was tentatively decided to use $M = 3$ and to restrict use of this formula to the case $A > 5$. For larger values of M the computation (35) is more time-consuming than the expansions discussed later in this section. With this M errors in φ_L and $\partial\varphi_L/\partial A$ are bounded by 0.00023. The formula for φ_L is explicitly

$$\varphi_L = \frac{0.66666667}{\sqrt{A^2 + B^2}} + \frac{1.2440169}{\sqrt{A^2 + (1.2679492 - B)^2}} + \frac{0.089316398}{\sqrt{A^2 + (4.7320508 - B)^2}} \qquad A > 5 \qquad (36)$$

and the formula for $\partial\varphi_L/\partial A$ is obtained by differentiating (36).

7.4 The Exponential-Integral Expansion for Small Values of the Ratio of Horizontal to Vertical Distance

For small values of the dimensionless horizontal distance A between the point source and the field point, the oscillating point-source potential is used in the form (28), and the potential φ_N from (29) is the one that must be evaluated. On page 477 of reference [4] two forms of the oscillating point-source potential are given. By comparing, these forms it is evident that in

TABLE 3

ABSCISSAS AND WEIGHTS FOR LAGUERRE-GAUSS QUADRATURE WITH ZERO VALUE OF THE FIRST ABSCISSA

j	t_j	H_j
M = 1		
1.0	0	1.0
M = 2		
1.0	0	0.5
2.0	2.0	0.5
M = 3		
1.0	0	0.33333333
2.0	1.26794919	0.62200847
3.0	4.73205081	0.04465820
M = 4		
1.0	0	0.25
2.0	0.93582223	0.62905268
3.0	3.30540729	0.11835639
4.0	7.75877048	0.00259093
M = 5		
1.0	0	0.20
2.0	0.74329193	0.60120469
3.0	2.57163501	0.18573233
4.0	5.73117875	0.01294285
5.0	10.95389431	0.00012013
M = 6		
1.0	0	0.16666667
2.0	0.61703085	0.56401481
3.0	2.11296596	0.23771357
4.0	4.61083315	0.03056192
5.0	8.39906697	0.00103820
6.0	14.26010307	0.00000484

TABLE 4

MAXIMUM ERRORS IN EVALUATING THE LINE SOURCE POTENTIAL φ_L USING LAGUERRE-GAUSS QUADRATURE WITH VARIOUS NUMBERS OF ABSCISSAS

	M = 1		M = 2		M = 3		M = 4		M = 5		M = 6	
A	φ_L	$\partial\varphi_L/\partial A$	φ_L	$\partial\varphi_L/\partial A$	φ_L	$\partial\varphi_L/\partial A$	φ_L	$\partial\varphi_L/\partial A$	φ_L	$\partial\varphi_L/\partial A$	φ_L	$\partial\varphi_L/\partial A$
2	0.14357	0.13188	0.03841	0.06015	0.01393	0.02606	0.00556	0.01115	0.00271	0.00609	0.00130	0.00319
3	0.07182	0.04715	0.01027	0.01203	0.00267	0.00393	0.00090	0.00153	0.00035	0.00065	0.00015	0.00032
4	0.04367	0.02209	0.00377	0.00342	0.00070	0.00086	0.00019	0.00027	0.00006	0.00010	0.00002	0.00004
5	0.02883	0.01203	0.00167	0.00127	0.00023	0.00023	0.00005	0.00006	0.00001	0.00002	0.00000	0.00001
6	0.02029	0.00722	0.00081	0.00055	0.00009	0.00008	0.00002	0.00002	0.00000	0.00000	0.00000	0.00000
8	0.01173	0.00317	0.00028	0.00013	0.00002	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
10	0.00757	0.00166	0.00012	0.00005	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

the present notation the following result holds

$$\frac{1}{r_1} - \nu\varphi_N = PV\int_0^\infty \frac{k+\nu}{k-\nu} e^{-(k/\nu)B} J_o\left(\frac{k}{\nu}A\right) dk \qquad (37)$$

where the designation PV signifies that the principal value of the integral is to be taken. With the variable change $k = \nu t$ and some rearrangement (37) yeilds for φ_N

$$\varphi_N = 2PV\int_0^\infty \frac{e^{-Bt}J_o(At)}{1-t} dt + \left[\frac{1}{\sqrt{A^2+B^2}} - \int_0^\infty e^{-Bt}J_o(At)dt\right] \qquad (38)$$

or

$$\varphi_N = 2PV\int_0^\infty \frac{e^{-Bt}J_o(At)}{1-t} dt \qquad (39)$$

This last follows from the fact that the bracketed term in (38) is zero as can be verified from a table of Laplace transforms, for example, page 1024 of reference [6]. The form (39) serves as the basis of an expansion involving the exponential integral Ei(B), which is defined as

$$Ei(B) = -PV\int_{-B}^\infty \frac{e^{-t}}{t} dt \qquad (40)$$

A simple variable change gives the result that

$$Q = PV\int_0^\infty \frac{e^{-Bt}}{1-t} dt = e^{-B}Ei(B) \qquad (41)$$

from which it immediately follows that

$$\frac{d^nQ}{dB^n} = (-1)^n PV\int_0^\infty \frac{t^n e^{-Bt}}{1-t} dt = \frac{d^n}{dB^n}\left[e^{-B}Ei(B)\right] \qquad (42)$$

To expand the integral in (39), replace J_o by its power series, which is convergent for all values of the argument. This gives

$$\varphi_N = 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{A}{2}\right)^{2k} PV \int_0^{\infty} \frac{t^{2k} e^{-Bt}}{1-t} \, dt \tag{43}$$

In view of the above this is

$$\varphi_N = 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{A}{2}\right)^{2k} \frac{d^{2k}}{dB^{2k}} \left[e^{-B} Ei(B)\right] \tag{44}$$

From (40)

$$\frac{d}{dB} [Ei(B)] = \frac{e^B}{B} \tag{45}$$

Repeated use of (45) in (44) gives the final form of the exponential-integral expansion. Specifically,

$$\varphi_N = 2 \sum_{k=0}^{\infty} (-1)^k C_k(B) \left(\frac{A}{B}\right)^{2k} \tag{46}$$

where

$$C_o(B) = e^{-B} Ei(B) \tag{47}$$

$$C_k(B) = \frac{B^{2k}}{2^{2k}(k!)^2} \left[e^{-B} Ei(B) - \sum_{\lambda=0}^{2k-1} \frac{\lambda!}{B^{\lambda+1}}\right] \tag{48}$$

The expansion for the A-derivative of φ_N is obtained from (40) by differentiation. The bracketed term in (48) consists of the difference between the quantity $e^{-B} Ei(B)$ and the first $2k$ terms in its asymptotic expansion for large B (see below).

The factor B^{2k} has been included in the coefficient functions given by (48) to minimize their variation with B. That this is the correct factor to accomplish this may be inferred from the limiting forms of these functions for very small and very large values of B. For large B the asymptotic series for the exponential integral gives

$$C_k(B) \to \frac{(2k-1)!}{2^{2k}(k!)^2}\frac{2k}{B}\left[1 + O\left(\frac{2k}{B}\right)\right], \qquad B \to \infty \tag{49}$$

For small B the form

$$C_k(B) \to -\frac{(2k-1)!}{2^{2k}(k!)^2}\left[1 + O(B)\right], \qquad B \to 0 \tag{50}$$

is obtained directly. Thus, $C_k(B)$ approaches zero as $1/B$ for large B and approaches a finite value for small B. Division of these quantities by B^{2k} would result in coefficient functions that rapidly approach zero for large B and that become large for small B. If the limiting forms (49) and (50) are used in the series (46), in both cases the ratio test shows convergence for $A/B < 1$ and divergence for $A/B > 1$.

Thus, the expansion for small values of A is actually an expansion in powers of A/B, and it is the value of this parameter that determines how many terms of the expansion must be used to obtain a certain desired accuracy. Clearly small values of A/B may correspond to rather large values of A if B is sufficiently large.

The approximate form of (46) that is used for computation is, of course,

$$\varphi_N = 2\sum_{k=0}^{M}(-1)^k C_k(B)\left(\frac{A}{B}\right)^{2k} \tag{51}$$

Values of φ_N calculated from (51) and of $\partial\varphi_N/\partial A$ calculated from the derivative of (51) have been compared with "brute force" numerical evaluations for various orders of the expansion, i.e., various values of M, over a wide range of values of B and A/B. For each value of A/B the maximum error for any value of B was determined and designated the maximum error at that

value of A/B. These maximum errors are shown in figure 11. (The functions themselves are shown in figures 9 and 10.) The errors in the derivative have been multiplied by A to minimize their dependence on B for fixed A/B. For $M = 4$ the expansion is accurate to 0.001 up to values of A/B of about 0.4 to 0.5. The smallest value of M that is used is $M = 1$. Notice that this means that two terms of the expansion (51) are used.

If subroutines are available for the exponential integral, the coefficient functions $C_k(B)$ can be calculated directly. However, such subroutines are rare. To fill this need, the functions $C_k(B)$ have been fitted with polynomials in the ranges: $0 \le B \le 1$, $1 \le B \le 2$, $2 \le B \le 4$, $4 \le B \le 8$, and $8 \le B \le 10$. The results are contained in Appendix A. All the $C_k(B)$ have individual polynomial fits, except $C_1(B)$, which is calculated from the relation

$$C_1(B) = \frac{1}{4}\left[B^2 C_o(B) - 1 - B\right] \tag{52}$$

which is obvious from the definition (48). This method of calculating $C_1(B)$ not only saves computing time, but also prevents numerical difficulty in the multipole expansion at small values of R or A (see section 8.3). The other coefficient functions can also be expressed as a multiple of $C_o(B)$ plus a polynomial, but the expressions contain $2k + 2$ terms and for the larger k are more complicated than the fitted polynomials. In Appendix A all the $C_k(B)$ are fitted to about the same accuracy, approximately 0.0001. When these functions are used, they are multiplied by $(A/B)^{2k}$, where A/B is never larger than one-half. Thus, less accuracy is required for the functions with larger values of k, and further investigation will very probably show that one or two terms may be omitted from the polynomial fits for $C_k(B)$, $k = 2, 3, 4$. This change will reduce computing time.

For $B > 10$, the coefficient functions are calculated using the asymptotic expansion of $e^{-B}Ei(B)$ for large B. This expansion is

$$e^{-B}Ei(B) \to \sum_{\lambda=0}^{\infty} \frac{\lambda!}{B^{\lambda+1}}, \qquad B \to \infty \tag{53}$$

The accuracy of an asymptotic expansion is usually optimized if the expansion is terminated with the smallest term. For $B = 10$, the tenth and eleventh terms of (53) are the smallest, but the accuracy is insensitive to term number between six and ten. For the present purposes six terms are used for C_0, C_2, and C_3, and eight terms are used for C_4. As mentioned above, use of the asymptotic expansion leads to some cancellation in the bracketed term in (48). The resulting expressions for the coefficient functions are:

$$C_0(B) \approx \frac{1}{B} + \frac{1}{B^2} + \frac{2}{B^3} + \frac{6}{B^4} + \frac{24}{B^5} + \frac{120}{B^6}$$

$$C_1(B) \approx \frac{1}{4}\left[B^2 C_0(B) - 1 - B\right]$$

$$C_2(B) \approx \frac{3}{8}\left[\frac{1}{B} + \frac{5}{B^2}\right] \qquad B > 10 \qquad (54)$$

$$C_3(B) \approx 0$$

$$C_4(B) \approx 0$$

The approximations (54) are less accurate for the higher values of k, but the lessening of accuracy with increasing k is consistent with the decreasing accuracy requirement that is discussed above.

7.5 Expansions for Finite Values of the Ratio of Vertical to Horizontal Distance

The expansion of the previous section calculates $\varphi_N(A,B)$ accurately for all values of the parameter B/A larger than some value in the range 2.0 to 2.5, say $B/A = 2.3$ for definiteness. What now remains to be devised is a method of calculating $\varphi_N(A,B)$ for the range $0 \le B/A \le 2.3$. It turns out that several expansions are required to do this, each of which is based on a particular value of B/A.

It is desired to expand $\varphi_N(A,B)$ about a particular value of B/A, which is denoted ρ, i.e., the expansion is about $B = \rho A$. All the expansions of this type are based on the formula

$$\varphi_N(A,B) = e^{-(B-\rho A)}\left[\varphi_N(A,\rho A) + 2\int_0^{(B/A)-\rho} \frac{e^{Au}\,du}{\sqrt{1+(u+\rho)^2}}\right] \tag{55}$$

which is obtained from (29) by straight-forward manipulation. First define

$$\epsilon = B - \rho A \tag{56}$$

The function $1/\sqrt{1+(u+\rho)^2}$ is expanded as a power series about $u = 0$ in the form

$$\frac{1}{\sqrt{1+(u+\rho)^2}} = \sum_{k=0}^{\infty} a_k(\rho)u^k \tag{57}$$

where the first few $a_k(\rho)$ are as follows

$$\begin{aligned}
\sqrt{1+\rho^2}\,a_0(\rho) &= 1 \\
\sqrt{1+\rho^2}\,a_1(\rho) &= -\frac{\rho}{1+\rho^2} \\
\sqrt{1+\rho^2}\,a_2(\rho) &= \frac{\rho^2 - 1/2}{(1+\rho^2)^2} \\
\sqrt{1+\rho^2}\,a_3(\rho) &= -\frac{\rho^3 - (3/2)\rho}{(1+\rho^2)^3} \\
\sqrt{1+\rho^2}\,a_4(\rho) &= \frac{\rho^4 - 3\rho^2 + 3/8}{(1+\rho^2)^4} \\
\sqrt{1+\rho^2}\,a_5(\rho) &= -\frac{\rho^5 - 5\rho^3 + (15/8)\rho}{(1+\rho^2)^5}
\end{aligned} \tag{58}$$

Now (55) becomes

$$\varphi_N(A,B) = e^{-\epsilon}\left[\varphi_N(A,\rho A) + 2\sum_{k=0}^{\infty} a_k(\rho)\int_0^{\epsilon/A} u^k e^{Au}du\right] \tag{59}$$

The integral in (59) is a standard form with value

$$\int_0^{\epsilon/A} u^k e^{Au}du = \frac{e^{\epsilon}}{A^{k+1}}\sum_{\lambda=0}^{k-1}(-1)^{\lambda}\frac{k!}{(k-\lambda)!}\epsilon^{k-\lambda} + \frac{(-1)^k k!}{A^{k+1}}(e^{\epsilon}-1) \tag{60}$$

This gives for φ_N

$$\varphi_N(A,B) = e^{-\epsilon}\left[\varphi_N(A,\rho A) + 2e^{\epsilon}\sum_{k=1}^{\infty}\frac{k!a_k(\rho)}{A^{k+1}}\sum_{\lambda=0}^{k-1}(-1)^{\lambda}\frac{\epsilon^{k-\lambda}}{(k-\lambda)!}\right.$$

$$\left. + 2(e^{\epsilon}-1)\sum_{k=0}^{\infty}(-1)^k\frac{k!}{A^{k+1}}a_k(\rho)\right] \tag{61}$$

This can be rearranged to give the final result

$$\varphi_N(A,B) \approx e^{-\epsilon}\varphi_N(A,\rho A) + \sum_{k=0}^{M}(-1)^k b_k(\rho) f_{k+2}(\epsilon)\left(\frac{\epsilon}{A}\right)^{k+1} \tag{62}$$

where

$$\sqrt{1+\rho^2}\, b_0(\rho) = 2$$

$$\sqrt{1+\rho^2}\, b_1(\rho) = \frac{\rho}{1+\rho^2}$$

$$\sqrt{1+\rho^2}\, b_2(\rho) = \frac{1}{3}\frac{2\rho^2-1}{(1+\rho^2)^2} \tag{63}$$

$$\sqrt{1+\rho^2}\, b_3(\rho) = \frac{1}{4}\frac{2\rho^3 - 3\rho}{(1+\rho^2)^3} \tag{63}$$

$$\sqrt{1+\rho^2}\, b_4(\rho) = \frac{1}{20}\frac{8\rho^4 - 24\rho^2 + 3}{(1+\rho^2)^4}$$

$$\sqrt{1+\rho^2}\, b_5(\rho) = \frac{1}{24}\frac{8\rho^5 - 40\rho^3 + 15\rho}{(1+\rho^2)^5}$$

and where the functions $f_k(\epsilon)$ are defined as

$$f_k(\epsilon) = \sum_{j=0}^{\infty} (-1)^j \frac{(k-1)!}{(k-1+j)!}\epsilon^j$$

$$= \frac{(k-1)!}{\epsilon^{k-1}}(-1)^{k-1}\left[e^{-\epsilon} - \sum_{\lambda=0}^{k-2}(-1)^\lambda \frac{\epsilon^\lambda}{\lambda!}\right]$$

$$= 1 - \frac{\epsilon}{k} + \frac{\epsilon^2}{k(k+1)} - \frac{\epsilon^3}{k(k+1)(k+2)} + \cdots \tag{64}$$

The derivative of φ_N is given by

$$\frac{\partial \varphi_N}{\partial A}(A,B) \approx e^{-\epsilon}\left\{\rho\varphi_N(A,\rho A) + \frac{d}{dA}\left[\varphi_N(A,\rho A)\right]\right\} + \frac{1}{A}\sum_{k=0}^{M+1}(-1)^k c_k(\rho) f_{k+1}(\epsilon)\left(\frac{\epsilon}{A}\right)^k \tag{65}$$

where

$$\sqrt{1+\rho^2}\, c_0(\rho) = -2\rho$$

$$\sqrt{1+\rho^2}\, c_1(\rho) = \frac{2}{1+\rho^2}$$

$$\sqrt{1+\rho^2}\, c_2(\rho) = 3\frac{\rho}{(1+\rho^2)^2} \tag{66}$$

$$\sqrt{1+\rho^2}\, c_3(\rho) = \frac{4\rho^2 - 1}{(1+\rho^2)^3} \tag{66}$$

$$\sqrt{1+\rho^2}\, c_4(\rho) = \frac{5}{4}\frac{4\rho^3 - 3\rho}{(1+\rho^2)^4}$$

$$\sqrt{1+\rho^2}\, c_5(\rho) = \frac{3}{4}\frac{8\rho^4 - 12\rho^2 + 1}{(1+\rho^2)^5}$$

Notice that the infinite series that would be required to give equality in (62) and (65) have been replaced by finite series for practical computation. The upper limit M of the summation for the potential is the order of the expansion. A total of $M + 1$ terms involving powers of ϵ are added to the basic term in the potential expansion (60), and $M + 2$ terms in powers of ϵ are added in the derivative expansion (63). (Both expansions are terminated with the same $f_k(\epsilon)$.)

If the expansions (62) and (65) are considered infinite series, the question of their convergence arises. From analysis of special cases and general order of magnitude arguments, it appears that the series convergence for $|\epsilon/A| < 1$.

The above formulas permit expansions up to $M = 4$ to be written down for any expansion parameter ρ. (In fact $M = 5$ is included for the potential expansion.) One value of ρ is special, namely $\rho = 0$. For this value, every other term in the expansion is zero, and for the same computational effort roughly twice as large a value of M can be used for $\rho = 0$ as for other values of ρ. Expansions have been worked out for three values of ρ: $\rho = 0$, $\rho = 1$, $\rho = 7/4$. These expansions are designated the near-zero, near-one, and near-seven-quarters expansions, respectively. They appear to cover the range $0 \le B/A \le 2.3$ adequately, but possibly a few small ranges require new expansions. For these values of ρ, the first few coefficients in the expansions are given in table 5. Specifically, table 5 contains $b_k(1)$, $c_k(1)$, $b_k(7/4)$, and $c_k(7/4)$ for $k = 0, 1, 2, 3, 4, 5$ and $b_k(0)$ and $c_k(0)$ for $k = 1, 2, 3, \ldots, 9$. With these coefficients the near-zero expansion can be written down with up to five nonzero terms in the summations — the

TABLE 5

COEFFICIENTS APPEARING IN THE EXPANSIONS FOR FINITE VALUES OF THE RATIO OF VERTICAL TO HORIZONTAL DISTANCE

K	$b_K(0)$	$c_K(0)$	$b_K(1)$	$c_K(1)$	$b_K(\frac{7}{4})$	$d_K(\frac{7}{4})$
0.0	2.0	0	1.41421356	-1.41421356	0.99227787	-1.73648628
1.0	0	2.0	0.35355339	0.70710678	0.21372138	0.24425301
2.0	-0.33333333	0	0.05892556	0.53033008	0.05135576	0.15782502
3.0	0	-1.0	-0.02209708	0.26516504	0.01011698	0.08324836
4.0	0.15	0	-0.02872621	0.05524271	0.00041268	0.03685696
5.0	0	0.75	-0.01565210	-0.04971844	-0.00106152	0.01320910
6.0	-0.08928571	0	-	-	-	-
7.0	0	0.625	-	-	-	-
8.0	0.06076388	0	-	-	-	-
9.0	0	-0.546875	-	-	-	-

same number as the other expansions. For $k > 5$ the general formulas for $b_k(\rho)$ and $c_k(\rho)$ are not given above, because these terms are used only for $\rho = 0$. For nonzero ρ it is more efficient computationally to derive more expansions than to extend existing expansions to high values of k.

The accuracy of the near-zero and near-one expansions for various orders M has been determined by comparing values of φ_N and $\partial\varphi_N/\partial A$ calculated by the expansions with "brute force" numerical evaluations for a considerable range of values of A and B/A. For each value of B/A the largest error that occurs for any A has been determined and designated the maximum error at that value of B/A. These maximum errors are shown in figures 12 and 13 for the near-zero and near-one expansions, respectively. The errors in the derivative have been multiplied by A to minimize their dependence on A for fixed B/A. An exhaustive error study has not been made for the near-seven-quarters expansion. However, from the form of the general expansion, it is felt that the errors for this expansion are nearly the same as those for the near-one expansion if the two expansions are compared at equal values of $\epsilon/A = B/A - \rho$. That is, figure 13 can serve approximately as the error plot for the near-seven-quarters expansion, if the abscissa scale is translated to make $7/4$ lie where 1 does now.

For small k the functions $f_k(\epsilon)$ are evaluated from the middle line of (64), i.e.,

$$f_1(\epsilon) = e^{-\epsilon}$$

$$\epsilon f_2(\epsilon) = 1 - e^{-\epsilon} \tag{67}$$

$$\epsilon^2 f_3(\epsilon) = 2[e^{-\epsilon} + \epsilon - 1]$$

It will be noticed that the functions $f_k(\epsilon)$ enter the expansions (62) and (65) multiplied by the powers of ϵ given in (67). For higher values of k, this method of evaluation is cumbersome. Accordingly, the functions $f_k(\epsilon)$ for $k = 3, 4, 5$ and 6 have been fitted with polynomials over a range $-2.5 \le \epsilon \le 2.5$. The functions $f_k(\epsilon)$ for $k = 8$, and 10, have been fitted over the range $0 \le \epsilon \le 2.5$. These latter functions are used only in the near-zero expansion for which ϵ cannot be negative. The polynomial fits of these functions are given in Appendix B.

To evaluate expansions (62) and (65) explicit formulas for the leading terms are required. Specifically, the functions

$$G(A,\rho) = \varphi_N(A,\rho A)$$

and

$$H(A,\rho) = \rho\varphi_N(A,\rho A) + \frac{d}{dA}\left[\varphi_N(A,\rho A)\right] \tag{68}$$

are needed in the range $0 \le A \le 5$. These two functions have been fitted with polynomials in the range $1 \le A \le 5$ for $\rho = 0$, 1, and $7/4$. The results are given in Appendix C. For the range $0 \le A \le 1$ improved polynomial fits were obtained by writing the above functions in the forms

$$G(A,\rho) = g(A,\rho) + 2e^{-\rho A}J_o(A)\ln\left(\frac{1}{2}A\right)$$

$$H(A,\rho) = h(A,\rho) - 2e^{-\rho A}J_1(A)\ln\left(\frac{1}{2}A\right) \tag{69}$$
$$+ 2e^{-\rho A}J_o(A)\frac{1}{A}$$

The functions $g(A,\rho)$ and $h(A,\rho)$ are not singular for small A, and it is these functions that are fitted with polynomials in the range $0 \le A \le 1$. The results are also given in Appendix C.

It can be seen from equation (69) that these expansions are singular for $A = 0$. With ρ fixed the condition $A \to 0$ implies that also $B \to 0$ and thus that the source, its image, and the field point approach each other and approach the free surface. Clearly, singularity is expected for this situation. Both these expansions and the multipole expansion of the following section fail for this condition.

If the expansions are used for values of A as large as five, the ranges of the fits in Appendix B are appropriate as long as $|\epsilon/A|$ is no greater than one-half. If it should prove necessary to use values of $|\epsilon/A|$ greater than one-half, either the range of the polynomial fits must be expanded, which is quite straightforward, or the expansions must be restricted to values of A less than some value less than five. In the latter case, the Laguerre-Gauss formulas must be used down to the new limit on A, and this can be arranged simply as can be seen in table 4. However, it is desirable to keep

values of $|\epsilon/A|$ as low as possible so that relatively few terms of the expansions (62) and (65) are needed for good accuracy.

7.6 A Sample Range Criterion for the Expansions

The formulas of the previous parts of this section can be used to calculate φ_N and its A-derivative to any degree of accuracy. The greater the accuracy that is required, the greater is the number of expansions that are needed, i.e., the greater is the number of values of ρ that must be used with (55). The accuracy requirement basically determines which expansion and how many terms of it are to be used at a particular set of values of A and B. However, the translation of this requirement into an actual logical decision routine for selecting the expansion and term number is a nontrivial task. To illustrate this procedure an example is presented here.

For illustrative purposes it is assumed that it is required to calculate φ_N and $\partial\varphi_N/\partial A$ with an error not exceeding 0.001 in absolute value. (Compare the values of these functions given in table 2 and figures 9 and 10.) This seems to be a reasonable error criterion for applications. Possibly, it is too conservative. (A decision routine for a different error criterion could be worked out from the error curves of figures 11, 12, and 13.) It is further assumed that only the expansions discussed in the previous parts of this section are to be used with only the terms that have been presented. The calculation is to cover the complete range of B and all A up to five. In all cases the fewest possible terms are to be used to conserve computing time.

The error curves shown in figures 11, 12, and 13 are used, but the results are adjusted, always in the conservative direction, to simplify the decision routine. The results are shown in figure 14, whose abscissa is A and whose ordinate is B/A. The coordinate plane is divided into regions, in each of which a certain number of terms of a particular expansion is used to calculate φ_N and $\partial\varphi_N/\partial A$. In all cases the number of terms is specified by M, where this quantity has the same meaning as it does in the definitions of the expansions that are given in the previous parts of this section. The error curves of the expansions have been conservatively simplified to make the boundaries of the regions in figure 14 straight lines, whose equations are given. It can be seen that the only regions that are not handled

satisfactorily are three small regions having small values of A near B/A values of 0.6, 1.4, and 2.1, respectively. These regions could be handled by means of one-term expansions about those values of ρ. Possibly an increase of the error criterion would cause these regions to disappear.

8.0 INTEGRATION OVER AN ELEMENT USING A MULTIPOLE EXPANSION

8.1 Geometric Quantities Associated with an Element

To carry out the method of solution outlined in section 6.5, it is necessary to develop formulas for calculating the potential and velocity at points in space due to a unit source density on one of the plane quadrilateral elements used to approximate the body surface (see figure 4). This requires integrating the formulas for the oscillating point source potential and its gradient over a general quadrilateral element. Certain geometric quantities associated with the quadrilateral are used to perform this integration. These quantities are defined in this section.

Figure 5 shows a general quadrilateral element. The coordinates of the centroid of the area of the quadrilateral are x_o, y_o, z_o. Integrations are performed in a coordinate system based on the element. The variables of this coordinate system are ξ, η, ζ, where the ξ- and η-axes lie in the plane of the element and the ζ-axis is normal to this plane. The origin of this coordinate system is taken as the centroid, and the unit vectors along the axes of the system are:

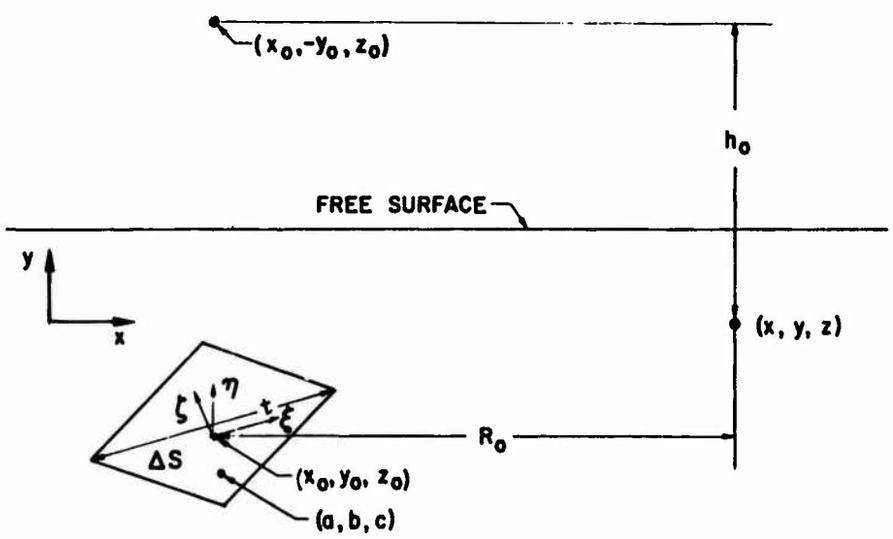

Figure 5. - Illustration of quantities used to obtain the multipole expansion.

$$\xi\text{-axis:} \quad a_{11}\vec{i} + a_{12}\vec{j} + a_{13}\vec{k}$$

$$\eta\text{-axis:} \quad a_{21}\vec{i} + a_{22}\vec{j} + a_{23}\vec{k} \tag{70}$$

$$\zeta\text{-axis:} \quad a_{31}\vec{i} + a_{32}\vec{j} + a_{33}\vec{k}$$

where $\vec{i}$, $\vec{j}$, $\vec{k}$ are the unit vectors along the x, y, z axes, respectively. A point with coordinates x, y, z has coordinates ξ, η, ζ in the system based on an element where these coordinates are related by

$$\begin{aligned} \xi &= a_{11}(x - x_o) + a_{12}(y - y_o) + a_{13}(z - z_o) \\ \eta &= a_{21}(x - x_o) + a_{22}(y - y_o) + a_{23}(z - z_o) \\ \zeta &= a_{31}(x - x_o) + a_{32}(y - y_o) + a_{33}(z - z_o) \end{aligned} \tag{71}$$

The maximum dimension of the quadrilateral is denoted t. Its value determines the size of the quadrilateral. The shape of the quadrilateral is expressed by the normalized moments of its area. Specifically, the normalized moments are:

$$I_{mn} = \frac{1}{t^{m+n+2}} \iint_{\Delta S} \xi^m \eta^n d\xi d\eta \tag{72}$$

where the integral is over the area ΔS of the element. The order of a particular moment is defined as the sum of its subscripts m+n. In general, the size of the moments decreases with increasing order. The zeroth order moment I_{00} is the normalized area of the element, and the second order moments, I_{20}, I_{11}, and I_{02}, are the normalized "moments of inertia". The first order moments, I_{10} and I_{01}, are zero because the origin of coordinates is the centroid of the area.

8.2 The Multipole Expansion

It will be recalled that the potential at a point (x,y,z) due to a unit oscillating point source at a point (a,b,c) is of the form

$$\varphi_s = \frac{1}{r} + \frac{1}{r_1} + \nu\varphi'(A,B) \tag{73}$$

In integrating this expression over an element the two $1/r$ terms are treated by the method of reference 1. The term of interest here is of one of two forms. Either

$$\nu\varphi'(A,B) = \nu[-\varphi_L(A,B) + \varphi_H(A,B)] \tag{74}$$

or

$$\nu\varphi'(A,B) = \nu[-\varphi_N(A,B) + \varphi_J(A,B,] \tag{75}$$

where the individual potentials φ_L, φ_H, φ_N, and φ_J are given by equations (26), (27), (29), and (30), respectively. In this section

$$\varphi = \varphi(A,B) \tag{76}$$

is used to denote any one of the four individual potentials of (74) and (75) which must be integrated over an element. The quantities A and B depend on the coordinates a,b,c of the point source. Specifically,

$$\begin{aligned} A &= \nu\sqrt{(x-a)^2 + (z-c)^2} = \nu R \\ B &= -\nu(y+b) = \nu h \end{aligned} \tag{77}$$

The process used to integrate the quantity $\nu\varphi$ over an element is known as the multipole expansion. In general terms it proceeds as follows. The point (a,b,c) where the source is located is taken as a general point on the quadrilateral element as shown in figure 5. This point is expressed in the element coordinate system by using a,b,c in place of x,y,z in equation (71). Because the point is on the element the resulting ζ is zero. By this means the quantities A and B, and thus the quantity $\nu\varphi$, are expressed in terms of the coordinates x_o, y_o, z_o of the centroid of the element and the

coordinates ξ, η of a point on the element in its own coordinate system. This expression for $\nu\varphi$ is expanded in a two-variable Taylor series in powers of ξ and η. The resulting series consists of certain derivatives of φ evaluated at the centroid, i.e., at the point $\xi = \eta = 0$, multiplied by powers of ξ and η. The derivatives of φ are constants with regard to integration over the element, and integration of the powers of ξ and η gives the normalized moments, equation (72).

The subscript 0 is used to denote quantities evaluated at the centroid, e.g., $(\partial\varphi/\partial A)_0$, etc. In particular

$$A_0 = \nu\sqrt{(x - x_o)^2 + (z - z_o)^2} = \nu R_o$$
$$B_0 = -\nu(y + y_o) = \nu h_o \tag{78}$$

Here R_o and h_o are, respectively, the horizontal and vertical distances between the field point (x,y,z) and the image point $(x_o,-y_o,z_o)$ of the centroid of the element.

In carrying out the details of the multipole expansion it turns out to be more efficient to expand the velocity components rather than the potential. Let the integrated potential of the element be

$$\Phi = \nu \iint_{\Delta S} \varphi d\xi d\eta \tag{79}$$

where φ may be any of the potentials φ_L, φ_H, φ_N, or φ_J. The velocity components associated with this potential are

$$V_x = -\nu \iint_{\Delta S} \frac{\partial\varphi}{\partial x} d\xi d\eta$$
$$V_y = -\nu \iint_{\Delta S} \frac{\partial\varphi}{\partial y} d\xi d\eta \tag{80}$$
$$V_z = -\nu \iint_{\Delta S} \frac{\partial\varphi}{\partial z} d\xi d\eta$$

These components are integrated over the element in the above-described manner. The potential can be expressed in terms of the y-component V_y. Specifically,

$$\Phi = -2\nu \iint_{\Delta S} \frac{d\xi d\eta}{r_1} - \frac{1}{\nu} V_y \qquad \text{if} \qquad \varphi = \varphi_L \quad \text{or} \quad \varphi_N$$

$$\Phi = -\frac{1}{\nu} V_y \qquad \text{if} \qquad \varphi = \varphi_H \quad \text{or} \quad \varphi_J \tag{81}$$

The integral term of (81) can be absorbed with the integral of the $1/r_1$ term of (73) and evaluated by the method of reference 1.

In deriving explicit formulas for the velocity components (80) use is made of the fact that the first order area moments, I_{10} and I_{01}, are zero. Use is also made of the fact that φ is an axisymmetric solution of Laplace's equation in cylindrical coordinates, i.e., that

$$\frac{1}{A} \frac{\partial}{\partial A} \left(A \frac{\partial \varphi}{\partial A} \right) + \frac{\partial^2 \varphi}{\partial B^2} = 0 \tag{82}$$

The manipulations required to derive the formulas of the multipole expansion are rather lengthy and are not included here. Instead only the results are presented.

The expansion for V_x is

$$
\begin{aligned}
V_x = {} & \left(\frac{\partial\varphi}{\partial B}\right)_0 (vt)^2 \left[b_{20}^{(x)} \right. \\
& - \left(\frac{\partial^2\varphi}{\partial B^2}\right)_0 (vt)^3 \left[b_{30}^{(x)} + b_{31}^{(x)} \left(\frac{t}{R_o}\right) + \ldots \right. \\
& + \left(\frac{\partial^3\varphi}{\partial B^3}\right)_0 (vt)^4 \left[b_{40}^{(x)} + \ldots \right. \\
& - \ldots \\
& + \ldots \\
& - \left(\frac{\partial\varphi}{\partial A}\right)_0 (vt)^2 \left[c_{20}^{(x)} + c_{21}^{(x)} \left(\frac{t}{R_o}\right) + c_{22}^{(x)} \left(\frac{t}{R_o}\right)^2 + \ldots \right] \\
& + \left(\frac{\partial^2\varphi}{\partial B \partial A}\right)_0 (vt)^3 \left[c_{30}^{(x)} + c_{31}^{(x)} \left(\frac{t}{R_o}\right) + \ldots \right. \\
& - \left(\frac{\partial^3\varphi}{\partial B^2 \partial A}\right)_0 (vt)^4 \left[c_{40}^{(x)} + \ldots \right.
\end{aligned}
\tag{83}
$$

The expressions for V_y and V_z are identical to the above with superscripts x replaced by superscripts y and z, respectively. Before explicit formulas for the various quantities in (83) are presented, some discussion of the general form of the expansion seems in order. The expansion (83) is the sum of two expansions in powers of (vt): the "b array" whose rows are multiplied by successive B-derivatives of φ, and the "c array" whose rows are multiplied by successive B-derivatives of $(\partial\varphi/\partial A)$. All rows of both arrays are infinite series in powers of (t/R_o), except the first row of the "b array", which consists of a single term. The complete expansion is thus in powers of the two variables (vt) and (t/R_o). The coefficients b_{mn} and c_{mn} (with any superscript x,y, or z) have subscripts that denote the powers of those variables that the coefficient multiplies. Specifically, the first subscript m

denotes the power of (vt), and the second subscript n denotes the power of (t/R_o). The coefficients b_{mn} and c_{mn} are each linear combinations of the normalized moments of order equal to two less than the sum of the subscripts of the coefficient. (That is, the sum of the subscripts of the coefficient is two greater than the sum of the subscripts of the normalized moments from which it is formed.) Thus, each coefficient b_{mn} or c_{mn} is designated as being of a particular order, which is simply the order of the normalized moments used to form it. The chief decrease in size of the coefficients in (83) is with increasing order. Note that coefficients of a fixed order are all on the same "antidiagonal" of the arrays in (83). This is similar to the multipole expansion described in reference [3].

In particular, the terms of (83) involving b_{20} and c_{20} are the zeroth order terms of the expansion. These coefficients depend on the normalized area I_{00} of the element. The first order coefficients b_{30}, c_{30}, and c_{21} depend on the first order moments I_{10} and I_{01}. Thus, these coefficients are zero if, as in the present case, the expansion is about the centroid of the element. They have been included in (83) to make clear the form of the expansion. The coefficients, b_{40}, b_{31}, c_{40}, c_{31}, and c_{22} are the second order coefficients. The expansion is used in two forms. Either the zeroth order terms are the only ones retained, or the zeroth order terms and all second order terms are retained. In the latter case, expansion (83) is used as written. Higher-order terms are not included because the complexity of the terms increases rapidly with order.

First define the auxiliary quantities

$$\alpha = \frac{x - x_o}{R_o}, \qquad \beta = \frac{y - y_o}{R_o}, \qquad \gamma = \frac{z - z_o}{R_o} \tag{84}$$

$$\bar{\alpha} = \frac{y}{R_o} - a_{12}\beta, \qquad \bar{\beta} = \frac{y}{R_o} - a_{22}\beta \tag{85}$$

$$K_1 = \bar{\alpha}\, I_{20} + \bar{\beta}\, I_{11}\,, \qquad K_2 = \bar{\alpha}\, I_{11} + \bar{\beta}\, I_{02} \tag{86}$$

$$J_1 = a_{12}K_1 + a_{22}K_2\,, \qquad J_2 = \bar{\alpha}\, K_1 + \bar{\beta}\, K_2 \tag{87}$$

$$G_1 = a_{12}I_{20} + a_{22}I_{11}\,, \qquad G_2 = a_{12}I_{11} + a_{22}I_{02} \tag{88}$$

$$H = a_{12}G_1 + a_{22}G_2 \tag{89}$$

$$J = 4J_2 + H - I_{20} - I_{02} \tag{90}$$

Then the coefficients in expansion (83) are as given in the tables below:

m	n	$b_{mn}^{(x)}$	$b_{mn}^{(y)}$	$b_{mn}^{(z)}$
2	0	0	I_{00}	0
3	0	0	0	0
3	1	$\left(\frac{1}{2}\alpha J - a_{11}K_1 - a_{21}K_2\right)$	0	$\left(\frac{1}{2}\gamma J - a_{13}K_1 - a_{23}K_2\right)$
4	0	αJ_1	$\frac{1}{2}(H - J_2)$	γJ_1

m	n	$c_{mn}^{(x)}$	$c_{mn}^{(y)}$	$c_{mn}^{(z)}$
2	0	αI_{00}	0	γI_{00}
2	1	0	0	0
2	2	$2b_{31}^{(x)}$	0	$2b_{31}^{(z)}$
3	0	0	0	0
3	1	$\left(2b_{40}^{(x)} - a_{11}G_1 - a_{21}G_2\right)$	$-\frac{1}{2}(J - 2J_2)$	$\left(2b_{40}^{(z)} - a_{13}G_1 - a_{23}G_2\right)$
4	0	$\alpha b_{40}^{(y)}$	$-J_1$	$\gamma b_{40}^{(y)}$

The many zeros in these tables greatly simplify the computation.

The B-derivatives in (83) can be simply related to the quantity that is differentiated. It is evident from definitions (27) and (30) or directly from relation (31) that

$$\frac{\partial^m \varphi}{\partial B^m} = (-1)^m \varphi$$

$$\frac{\partial^{m+1} \varphi}{\partial B^m \partial A} = (-1)^m \frac{\partial \varphi}{\partial A} \qquad \text{if} \qquad \varphi = \varphi_H \quad \text{or} \quad \varphi_J \tag{91}$$

For φ_L and φ_N the B-derivatives can be computed successively from the definitions (26) and (29) or more directly from relation (32). The results are as follows:

if $\varphi = \varphi_L$ or φ_N

$$\frac{\partial \varphi}{\partial B} = -\left[\varphi - \frac{2}{\nu r_1}\right]$$

$$\frac{\partial^2 \varphi}{\partial B^2} = +\left[\varphi - \frac{2}{\nu r_1} - \frac{2B}{(\nu r_1)^3}\right]$$

$$\frac{\partial^3 \varphi}{\partial B^3} = -\left[\varphi - \frac{2}{\nu r_1} - \frac{2B}{(\nu r_1)^3} + \frac{2}{(\nu r_1)^3} - \frac{6B^2}{(\nu r_1)^5}\right] \tag{92}$$

$$\frac{\partial^2 \varphi}{\partial B \partial A} = -\left[\frac{\partial \varphi}{\partial A} + \frac{2A}{(\nu r_1)^3}\right]$$

$$\frac{\partial^3 \varphi}{\partial B^2 \partial A} = \left[\frac{\partial \varphi}{\partial A} + \frac{2A}{(\nu r_1)^3} + \frac{6AB}{(\nu r_1)^5}\right]$$

where as before

$$\nu r_1 = \sqrt{A^2 + B^2} \tag{93}$$

Expressions similar to (92) could be derived for the higher derivatives, so that in principle expansion (83) can be carried to any desired order. The

derivatives in (92) are all that are needed if expansion (83) is terminated with second order terms.

The formulas of this section permit the potential and velocity at a general point in space due to a general quadrilateral element to be evaluated in terms of the potential and A-derivative of the potential due to a point-source at the centroid of the quadrilateral. These two quantities are evaluated by the formulas of section 7.0. Thus, the multipole expansion reduces the quadrilateral source to a point source as far as computation is concerned and is effective for all range of A and B.

8.3 Behavior of the Multipole Expansion at Small Values of Horizontal Distance

From the form of expansion (83) it appears that the expansion becomes singular as R_o/t approaches zero, i.e., as the horizontal distance between the centroid of the element and the field point becomes small. However, it is clear that if the potential φ is an analytic function, as are all φ considered for the present purpose, this singularity must be apparent and not real. It is possible that the apparent singularity might lead to numerical difficulty, and calculational procedures must be designed to avoid such problems. It turns out that to avoid any singularity the multipole expansion must include either all terms of a particular order or no such terms. The procedure adopted in this report is such that the apparent singularity causes no difficulty. Of course, expansion (83) cannot be used if R_o/t is exactly zero. Thus, a test is made and special limiting formulas are used if $|R_o/t| < \epsilon$, where ϵ is taken as, say, 10^{-4}. The discussion below shows that the expansion formula may be used for small values of R_o/t just greater than any nonzero ϵ.

The potentially troublesome terms of (83) are those containing negative powers of R_o/t. For the present discussion it suffices to consider only

those terms and to ignore the others. The sum of the singular terms of (83) are (dropping the subscripts o)

$$V_x(\text{sing.}) = -\left(\frac{\partial^2\varphi}{\partial B^2}\right)(\nu t)^3\, b_{31}^{(x)}\left(\frac{t}{R}\right) - \left(\frac{\partial\varphi}{\partial A}\right)(\nu t)^2\, c_{22}^{(x)}\left(\frac{t}{R}\right)^2 + \left(\frac{\partial\varphi}{\partial B\partial A}\right)(\nu t)^3\, c_{31}^{(x)}\left(\frac{t}{R}\right) \tag{94}$$

For finite values of ν and $B = \nu h$ taking the limit as $R \to 0$ means that $A = \nu R$ and thus A/B are both small. Thus, the potential is considered only in the form (28), i.e., φ_N and φ_J are used in expansion (83), not φ_L and φ_H. Also, the potential φ_N and its A-derivative are evaluated by means of the exponential-integral expansion using an order of the expansion $M = 1$ (figure 14). Thus, equation (51) gives for the potential and its derivative

$$\varphi_N = 2\left[C_o(B) - C_1(B)\left(\frac{R}{h}\right)^2\right] + O(R^4)$$
$$\frac{\partial\varphi_N}{\partial A} = \left[-\frac{4}{B}\,C_1(B)\left(\frac{R}{h}\right)\right] + O(R^3) \tag{95}$$

where the $O(R^4)$ and $O(R^3)$ terms in (95) merely illustrate the magnitude of neglected terms, since only the terms in square brackets are retained for $M = 1$. The coefficient $C_1(B)$ is not calculated independently but is obtained from $C_o(B)$ by equation (52). Thus, the above expressions are

$$\varphi_N = C_o(B)\left[2 - \frac{1}{2}(\nu R)^2\right] + \frac{B+1}{2}\left(\frac{R}{h}\right)^2 + O(R^4)$$
$$\frac{\partial\varphi_N}{\partial A} = -C_o(B)(\nu R) + \frac{B+1}{B}\left(\frac{R}{h}\right) + O(R^3) \tag{96}$$

If (96) and (92) are used in (94), the result is (dropping the $O(R^4)$ and $O(R^3)$ terms that contribute nothing in the limit)

$$V_x(\text{sing.}) = -\left\{C_o(B)\left[2 - \frac{1}{2}(\nu R)^2\right] + \frac{B+1}{2}\left(\frac{R}{h}\right)^2 - \frac{2}{\nu r_1} - \frac{2B}{(\nu r_1)^3}\right\} \times (\nu t)^3 b_{31}^{(x)}\left(\frac{t}{R}\right)$$
$$+ \left\{C_o(B)(\nu R) - \frac{B+1}{B}\left(\frac{R}{h}\right)\right\}(\nu t)^2 C_{22}^{(x)}\left(\frac{t}{R}\right)^2$$
$$+ \left\{C_o(B)(\nu R) - \frac{B+1}{B}\left(\frac{R}{h}\right) - \frac{2(\nu R)}{(\nu r_1)^3}\right\}(\nu t)^3 C_{31}^{(x)}\left(\frac{t}{R}\right) \tag{97}$$

or

$$V_x(\text{sing.}) = \left\{-\left[2C_o(B) + \frac{2}{\nu r_1} + \frac{2B}{(\nu r_1)^3}\right](\nu t)^3 b_{31}^{(x)}\left(\frac{t}{R}\right)\right\}$$
$$+ \left\{\left[C_o(B) - \frac{B+1}{B^2}\right](\nu t)^3 c_{22}^{(x)}\left(\frac{t}{R}\right)\right\} \tag{98}$$
$$+ c_{31}^{(x)}(\nu t)^4\left[C_o(B) - \frac{B+1}{B^2} - \frac{2}{(\nu r_1)^3}\right] + O(R)$$

The two terms in curly brackets in (98) will be computed separately in practice. Both terms have singularities of order $1/R$ with coefficients that are the terms in square brackets. These singularities, which are shown below to cancel analytically, must also cancel numerically. Since the singularity is only first order, this can be arranged with moderate caution in the calculational procedure. This has been verified explicitly by numerical experimentation at $R_o/t = \epsilon = 10^{-4}$.

There are two kinds of terms in (98). The terms that do not involve $C_o(B)$ are exact except for round-off caused by the finite word length of the computer. The terms containing $C_o(B)$ have an additional, much larger

error due to the polynomial fit of $C_o(B)$. To show these two types of terms explicitly, (98) is written in the form

$$V_x(\text{sing.}) = (\nu t)^3 \left(\frac{t}{R}\right)\left\{C_o(B)\left[-2b_{31}^{(x)} + c_{22}^{(x)}\right] + \left[2b_{31}^{(x)}\left(\frac{1}{\nu r_1} + \frac{B}{(\nu r_1)^3}\right) - c_{22}^{(x)}\frac{B+1}{B^2}\right]\right\} \tag{99}$$

$$+ (\nu t)^4 c_{31}^{(x)}\left[C_o(B) - \frac{B+1}{B^2} - \frac{2}{(\nu r_1)^3}\right] + O(R)$$

From the table of coefficient functions above, it can be seen that

$$c_{22}^{(x)} = 2b_{31}^{(x)} \tag{100}$$

and the same statement is true if the superscript x is replaced by y or z. Thus, the coefficient of $(t/R)C_o(B)$ in (99) vanishes, and thus errors due to fitting $C_o(B)$ are not critical. This is entirely due to the fact that $C_1(B)$ is computed in terms of $C_o(B)$ from (52). If $C_1(B)$ were computed separately, $C_o(B)$ and $C_1(B)$ would have independent errors and the required cancellation would not occur. There would then be numerical difficulty for small R. To show the disappearance in the limit of the second term in (99) it should be noticed that

$$\frac{1}{\nu r_1} = \frac{1}{\sqrt{A^2 + B^2}} = \frac{1}{B} + O(R^2) \tag{101}$$

This, together with (100) gives the desired result. Thus, in the limit as $R \to 0$

$$V_x(\text{sing.}) \to (\nu t)^4 c_{31}^{(x)}\left[C_o(B) - \frac{B+3}{B^2}\right] \tag{102}$$

with similar expressions for V_y and V_z. These expressions may be used for $\left|R_o/t\right| < \epsilon$.

Methods of computing the potential φ_J, which is given by (30), are not discussed in section 7.0, because it is felt that standard expressions will be adequate. In particular, for moderate-to-small values of A, it seems reasonable to use the rapidly convergent Taylor series for $J_o(A)$ in powers of A. In this case the apparent singularities in the multipole expansion can be shown to cancel analytically and numerical difficulty at small R should be no worse than for φ_N.

8.4 Accuracy of the Multipole Expansion

The accuracy of the multipole expansion was investigated by using it to calculate a large number of cases and comparing the results with the results of "brute force" numerical integrations. In formulating error curves one problem was to be conservative without being too conservative. The procedure used to generate these error curves is described below.

The potential was considered in the form (28), i.e., divided into φ_N and φ_J. (The $1/r$ terms were not considered.) Thus the potential was divided into its real part $-v\varphi_N$ and its imaginary part $v\varphi_J/i$. In the derivative expressions (91) and (92) that enter into the expansion (83) exact values of φ and $(\partial\varphi/\partial A)$ were used, so that any errors are those associated with the multipole expansion. The accuracy of the multipole expansion depends on the parameters (h_o/t), (R_o/t) and (vt) and also on the shape of the element and on the direction of the field point with respect to the centroid of the element. In calculating error curves, the maximum error with respect to element shape, direction, and velocity component was determined as a function of the three parameters above. Specifically, a set of values of (h_o/t), (R_o/t), and (vt) was selected. Calculations were performed for six different elements and for a variety of directions. The largest error in any velocity component for any of the elements and directions was determined by inspection. This was done for the potentials $v\varphi_N$ and $v\varphi_J$ separately to obtain a "maximum real part" and a "maximum imaginary part" error. The square root of the sum of the squares of these two errors is the quantity eventually plotted. It has the nature of a "maximum absolute value" error, and it is conservative because the "maximum real part" and "maximum imaginary part" errors seldom occur for the same velocity component, the same element, and

the same direction. The above procedure was repeated for ranges of values of (h_o/t), (R_o/t), and (νt) to obtain the maximum error curves presented here.

Maximum error curves were calculated for the case where all second-order multipole terms were retained, i.e., where expansion (83) is used as written, and also for the case where only zeroth-order multipole terms are retained, i.e., only the b_{20} and c_{20} terms of (83). Figure 15 shows error curves for the second-order multipole expansion, and figure 16 shows error curves for the zeroth-order multipole expansion. Values of (νt) up to three were considered. For $(\nu t) = 3$ the maximum element dimension is approximately equal to one-half the wave length of the motion. No higher values of (νt) were considered, because it was felt that at such frequencies the assumption of a constant source density on the element would break down. If, however, this assumption remains valid, higher frequencies can be considered by subdividing elements to reduce the maximum value of (νt) below three (see next section). A comparison of figures 15 and 16 for $(h_o/t) = 2$ and $(h_o/t) = 3$ shows the very large improvement in accuracy that is obtained by using the second-order multipole expansion instead of the zeroth order. This fact was also a factor in the decision not to proceed to higher orders for the multipole expansion.

If a definite error criterion is adopted, figures 15 and 16 permit ranges of validity to be established for the zeroth and second-order multipole expansions. For definiteness, it is assumed here that an absolute error criterion of 0.001 has been adopted. With this criterion the ranges may be defined:

a. For $(h_o/t) > 4$, zeroth-order multipole is sufficient for all (R_o/t) and all $(\nu t) \leq 3$. (As can be seen from figure 16c, possibly $(h_o/t) > 4.2$ is required for small (R_o/t), but this is a detail.)

b. For $4 > (h_o/t) > 2$, second-order multipole gives the required accuracy for all (R_o/t) and all $(\nu t) \leq 3$.

c. For $(h_o/t) < 2$ second-order multipole is not sufficiently accurate for all (R_o/t) and all $(\nu t) \leq 3$. However, the accuracy improves very rapidly with decreasing (νt) and with increasing (R_o/t).

In the range $(h_o/t) < 2$ the integration over an element is accomplished by subdividing the element into subelements, each of which has a smaller value of t. Thus, when calculating the effect of the subelement, the effective value of (vt) is reduced compared to that of the original element. In most cases the values of (h_o/t) and (R_o/t) are increased relative to those of the original element. As can be seen from figure 15, all three of these changes in parameter values lead to a reduction of the error for the second-order multipole. The computation of the effect on itself of an element adjacent to the free surface may require special handling. This is acceptable from the standpoint of computing time, because such cases represent a small fraction of the total computation. However, the use of a second-order multipole with element subdivision will give accurate results for the important case of the effect of one element adjacent to the free surface on another such element. It is only for elements near the free surface that subdivision is required, and thus it is used rather infrequently in the computation scheme.

8.5 A Scheme for Subdividing an Element

The subdivision of an element for use with the multipole expansion at values of (h_o/t) less than two may be accomplished in a variety of ways. From a computational standpoint the process of subdividing an element consists of the calculation of the sets of geometrical quantities that define the subelements. Specifically, the following quantities must be calculated for each subelement: the normalized moments I_{mn} of the area, the coordinates of the centroid, the maximum dimension t, and the components a_{11}, a_{21}, etc. of the unit vectors along the axes of the subelements' coordinate systems. (In reference 1 the set of these a_{mn}'s is called the transformation matrix.) If further subdivision is required to reduce values of (vt) to acceptable values, i.e., if the subelements must themselves be subdivided, the coordinates ξ_k, η_k, $k = 1,2,3,4$ of the corners of the subelements in their own coordinate systems must also be calculated. An efficient subdivision scheme is one that: (1) obtains all the above quantities with as little computation as possible, (2) reduces values of (vt) as much as possible, and (3) can be iterated in a straightforward manner to subdivide the subelements. Many schemes are possible. One scheme that possesses these requirements is outlined here.

A typical element to be subdivided is shown in figure 6. The coordinates of its four corner points in the element coordinate system are ξ_k, η_k, $k = 1,2,3,4$. It is assumed that the maximum dimension t of the element is taken as the longer of its two diagonals and that the ξ-axis of the element coordinate system is taken parallel to this longer diagonal. Thus, if (as shown in figure 6) the longer diagonal is between the points (ξ_1,η_1) and (ξ_3,η_3), then $\eta_1 = \eta_3$ and $t = (\xi_3 - \xi_1)$. The origin of the element coordinate system is taken as the centroid of the element. (All of the above is consistent with the geometrical considerations of reference 1. The only required change is making the ξ-axis parallel to the longer diagonal.)

Conceptually, the subdivision scheme consists of bisecting each side of the quadrilateral with a point and drawing a line connecting each of these four points with the midpoint of the longer diagonal. As shown in figure 6, this process yields four subelements, which are labelled 1, 2, 3, 4 to denote the corner point they contain. Subelements 1 and 3 have the same shape

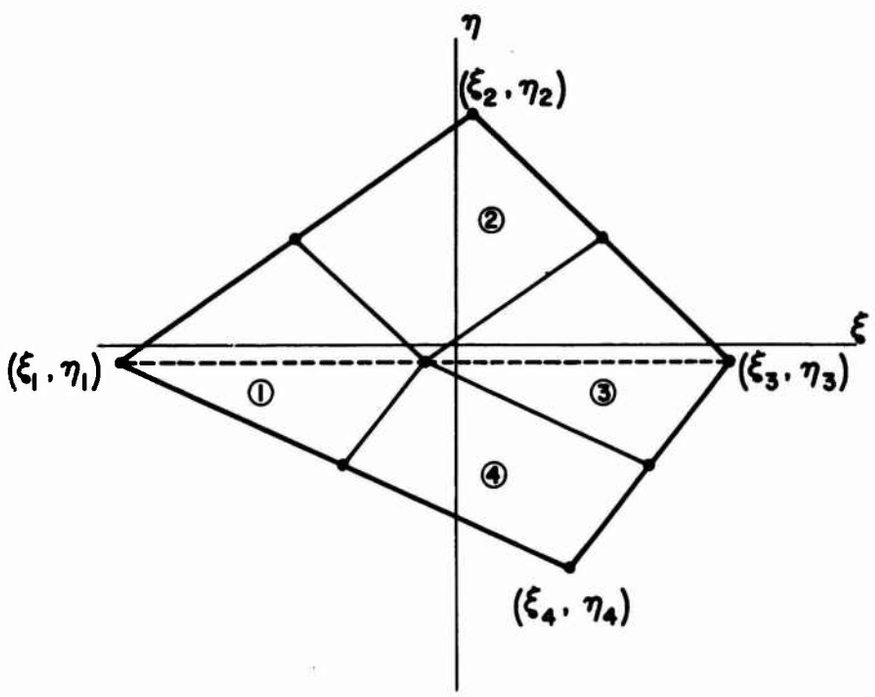

Figure 6. - Subdivision of an element by the use of midpoints.

and orientation as the original element, and each is reduced by one-half in linear dimension. Subelements 2 and 4 are parallelograms. Each subelement has one diagonal parallel to the longer diagonal of the original element. The ξ-axes of the coordinate systems of all subelements are thus taken parallel to the ξ-axis of the coordinate system of the original element. Thus, the components a_{11}, a_{21}, etc., of the unit vectors along the axes of the coordinate systems of all subelements are identical with the same quantities for the original element, and no additional computation is required for these quantities. Subelements 1 and 3 have the same normalized moments as the original element and have values of t exactly half that of the original element. Normalized moments must be computed for the parallelogram subelements 2 and 4, but in subsequent subdivision of the parallelograms the normalized moments are identical for all later subelements. The length of the longer diagonal t of one of the parallelogram subelements cannot be predicted in advance. The diagonal parallel to the ξ-axis is exactly half the value of t for the original element, but the other diagonal may be longer. In the worst case the other diagonal of a parallelogram subelements may be only slightly shorter than the value of t for the original element. However, in subsequent subdivisions, the values of t for the subelements of the parallelogram are exactly half the value of t for the parallelogram.

Explicit formulas for the geometric quantities associated with the subelements 1, 2, 3, and 4 of figure 6 show how little computation is required by the subdivision process. First define

$$m_2 = \frac{\eta_2 - \eta_1}{\xi_3 - \xi_1}, \qquad m_4 = \frac{\eta_1 - \eta_4}{\xi_3 - \xi_1}$$

$$e_2 = \frac{2\xi_2 - \xi_1 - \xi_3}{\xi_3 - \xi_1}, \qquad e_4 = \frac{2\xi_4 - \xi_1 - \xi_3}{\xi_3 - \xi_1} \tag{103}$$

Then the geometric quantities associated with the subelements in figure 6 are as follows:

Element	Maximum Diagonal	ξ(Centroid)	η(Centroid)	I_{00}
Original	t	0	0	I_{00}
Subelement 1	$\frac{1}{2}$ t	$\frac{1}{2}\xi_1$	$\frac{1}{2}\eta_1$	I_{00}
Subelement 2	see above	$\frac{1}{4}(\xi_1 + 2\xi_2 + \xi_3)$	$\frac{1}{2}(\eta_1 + \eta_2)$	m_2
Subelement 3	$\frac{1}{2}$ t	$\frac{1}{2}\xi_3$	$\frac{1}{2}\eta_1$	I_{00}
Subelement 4	see above	$\frac{1}{4}(\xi_1 + 2\xi_4 + \xi_3)$	$\frac{1}{2}(\eta_1 + \eta_4)$	m_4

Element	I_{20}	I_{11}	I_{02}
Original	I_{20}	I_{11}	I_{02}
Subelement 1	I_{20}	I_{11}	I_{02}
Subelement 2	$\frac{1}{24} m_2(e_2^2 + 1)$	0	$\frac{1}{6} m_2^3$
Subelement 3	I_{20}	I_{11}	I_{02}
Subelement 4	$\frac{1}{24} m_4(e_4^2 + 1)$	0	$\frac{1}{6} m_4^3$

Finally, from the method of subdivision it is clear that the coordinates of the corner points of the subelements of figure 6 are simple averages of the coordinates of the corner points of the original element.

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APPENDIX A

EXPRESSIONS FOR EVALUATING THE COEFFICIENT FUNCTIONS IN THE EXPONENTIAL-INTEGRAL EXPANSION FOR VALUES OF THE ARGUMENT FROM ZERO TO TEN

$C_o(B) = e^{-B} Ei(B)$

$0 \leq B \leq 1$

$$C_o(B) = e^{-B} \ln B + 0.57723459 + 0.42188715\,B - 0.45420044\,B^2 + 0.18830916\,B^3 - 0.036074946\,B^4 + E(B)$$

$|E(B)| < 25 \cdot 10^{-6}$

$E(1) = -0.000019$

$1 \leq B \leq 2$

$$C_o(B) = 0.69717880 + 0.30182470\,(B-1) - 0.63786604\,(B-1)^2 + 0.48345948\,(B-1)^3 - 0.22434376\,(B-1)^4 + 0.050233747\,(B-1)^5 + E(B)$$

$|E(B)| < 20 \cdot 10^{-6}$

$E(1) = 0.000004$

$E(2) = 0.000004$

$2 \leq B \leq 4$

$$C_o(B) = 0.67048886 - 0.17123280\,(B-2) - 0.034727768\,(B-2)^2 + 0.042573269\,(B-2)^3 - 0.014448463\,(B-2)^4 + 0.0019073418\,(B-2)^5 + E(B)$$

$|E(B)| < 25 \cdot 10^{-6}$

$E(2) = 0.000006$

$E(4) = 0.000006$

$4 \leq B \leq 8$

$$C_o(B) = 0.35953823 - 0.10948767\,(B-4) + 0.023664245\,(B-4)^2 - 0.0031370348\,(B-4)^3 + 0.00018870981\,(B-4)^4 + E(B)$$

$|E(B)| < 30 \cdot 10^{-6}$

$E(4) = -0.000014$

$E(8) = -0.000014$

$8 \leq B \leq 10$

$$C_o(B) = 0.14772609 - 0.022641303\,(B-8) + 0.0033149587\,(B-8)^2 - 0.000320112\,96\,(B-8)^3 + E(B)$$

$|E(B)| < 15 \cdot 10^{-6}$

$E(8) = -0.000005$

$E(10) = -0.000005$

$$C_1(B) = \frac{1}{4}\,[B^2 e^{-B} Ei(B) - 1 - B]$$

In all ranges of B $C_1(B)$ is calculated from the identity

$$C_1(B) = \frac{1}{4}\,[B^2 C_o(B) - 1 - B]$$

$$C_2(B) = \frac{1}{64}\,[B^4 e^{-B} Ei(B) - B^3 - B^2 - 2B - 6]$$

$0 \leq B \leq 1$

$$C_2(B) = -0.093764999 - 0.030660348\,B - 0.018986087\,B^2 - 0.016060631\,B^3 + 0.014132838\,B^4 + E(B)$$

$|E(B)| < 20 \cdot 10^{-6}$

$E(1) = 0.000017$

$1 \leq B \leq 2$

$$C_2(B) = -0.14535703 - 0.061019814\,(B-1) + 0.011051671\,(B-1)^2 + 0.026504571\,(B-1)^3 - 0.0073085699\,(B-1)^4 + E(B)$$

$|E(B)| < 10 \cdot 10^{-6}$

$E(1) = 0.000000$

$E(2) = 0.000000$

$2 \leq B \leq 4$

$$C_2(B) = -0.17609428 + 0.010002509\,(B-2) + 0.052699467\,(B-2)^2 - 0.010659030\,(B-2)^3 + E(B)$$

$|E(B)| < 75 \cdot 10^{-6}$

$E(2) = 0.000035$

$E(4) = -0.000022$

$4 \leq B \leq 8$

$$C_2(B) = -0.030565742 + 0.094174067\,(B-4) - 0.0089971945\,(B-4)^2 - 0.0055536427\,(B-4)^3 + 0.0014851919\,(B-4)^4 - 0.00011322320\,(B-4)^5 + E(B)$$

$|E(B)| < 35 \cdot 10^{-6}$

$E(4) = -0.000024$

$E(8) = -0.000023$

$8 \leq B \leq 10$

$$C_2(B) = 0.11105052 - 0.0089389738\,(B-8) - 0.0034737151\,(B-8)^2 + 0.00077624732\,(B-8)^3 + E(B)$$

$|E(B)| < 25 \cdot 10^{-6}$

$E(8) = 0.000017$

$E(10) = 0.000016$

$$C_3(B) = \frac{1}{2304}\left[B^6 e^{-B} Ei(B) - B^5 - B^4 - 2B^3 - 6B^2 - 24B - 120\right]$$

$0 \leq B \leq 1$

$$C_3(B) = -0.052081517 - 0.010485777\,B - 0.0021508276\,B^2 - 0.0018207172\,B^3 + E(B)$$

$|E(B)| < 10 \cdot 10^{-6}$

$E(1) = -0.000001$

$1 \leq B \leq 2$

$$C_3(B) = -0.066549269 - 0.019776277\,(B-1) - 0.0092101808\,(B-1)^2 + 0.0030382825\,(B-1)^3 + E(B)$$

$|E(B)| < 20 \cdot 10^{-6}$

$E(1) = -0.000012$

$E(2) = -0.000011$

$2 \le B \le 4$

$$C_3(B) = -0.092491866 - 0.028463143\,(B-2) + 0.00033224846\,(B-2)^2 + 0.0074613867\,(B-2)^3 - 0.0011826565\,(B-2)^4 + E(B)$$

$|E(B)| < 20 \cdot 10^{-6}$

$\left|\begin{matrix} E(2) = -0.000005 \\ E(4) = 0.000004 \end{matrix}\right|$

$4 \le B \le 8$

$$C_3(B) = -0.10729389 + 0.023850510\,(B-4) + 0.018257392\,(B-4)^2 - 0.0039381087\,(B-4)^3 - 0.00001436649\,(B-4)^4 + 0.000036616026\,(B-4)^5 + E(B)$$

$|E(B)| < 40 \cdot 10^{-6}$

$\left|\begin{matrix} E(4) = 0.000030 \\ E(8) = 0.000028 \end{matrix}\right|$

$8 \le B \le 10$

$$C_3(B) = 0.061968114 + 0.023687746\,(B-8) - 0.0086533615\,(B-8)^2 + 0.00080471344\,(B-8)^3 + E(B)$$

$|E(B)| < 25 \cdot 10^{-6}$

$\left|\begin{matrix} E(8) = -0.000009 \\ E(10) = -0.000004 \end{matrix}\right|$

$$C_4(B) = \frac{1}{147,456}\left[B^8 e^{-B} Ei(B) - B^7 - B^6 - 2B^5 - 6B^4 - 24B^3 - 120B^2 - 702B - 5040\right]$$

$0 \le B \le 1$

$$C_4(B) = -0.034191950 - 0.0047257587\,B - 0.0011700246\,B^2 + E(B)$$

$|E(B)| < 20 \cdot 10^{-6}$

$\left|E(1) = 0.000014\right|$

$1 \le B \le 2$

$$C_4(B) = -0.040121243 - 0.0069648968\,(B-1) - 0.0026223112\,(B-1)^2 + E(B)$$

$|E(B)| < 25 \cdot 10^{-6}$

$\left|\begin{matrix} E(1) = -0.000019 \\ E(2) = +0.000017 \end{matrix}\right|$

$2 \leq B \leq 4$

$$C_4(B) = -0.049763691 - 0.011747816\,(B-2) - 0.0050354531\,(B-2)^2 + 0.0016034521\,(B-2)^3 + E(B)$$

$|E(B)| < 50 \cdot 10^{-6}$

$E(2) = -0.000038$

$E(4) = -0.000032$

$4 \leq B \leq 8$

$$C_4(B) = -0.080550890 - 0.011972214\,(B-4) + 0.0048073235\,(B-4)^2 + 0.0027910827\,(B-4)^3 - 0.00072846533\,(B-4)^4 + 0.000046982416\,(B-4)^5 + E(B)$$

$|E(B)| < 20 \cdot 10^{-6}$

$E(4) = -0.000009$

$E(8) = -0.000005$

$8 \leq B \leq 10$

$$C_4(B) = -0.011289474 + 0.034544199\,(B-8) - 0.0023406480\,(B-8)^2 - 0.00057919267\,(B-8)^3 + E(B)$$

$|E(B)| < 30 \cdot 10^{-6}$

$E(8) = -0.000024$

$E(10) = -0.000021$

APPENDIX B

POLYNOMIAL FITS FOR THE COEFFICIENT FUNCTIONS APPEARING IN THE EXPANSIONS FOR FINITE VALUES OF THE RATIO OF VERTICAL TO HORIZONTAL DISTANCE

$-2.5 \leq \epsilon \leq -1.6$

$$f_3(\epsilon) = 1.8383074 - 0.79033419\,(\epsilon + 1.6) + 0.22952184\,(\epsilon + 1.6)^2 - 0.047541147\,(\epsilon + 1.6)^3 + 0.012524009\,(\epsilon + 1.6)^4 + E(\epsilon)$$

$|E(\epsilon)| < 10 \cdot 10^{-6}$

$E(-1.6) = 0.000001$

$-1.6 \leq \epsilon \leq 0$

$$f_3(\epsilon) = 1.0000131 - 0.33290257\,\epsilon + 0.085419775\,\epsilon^2 - 0.013188412\,\epsilon^3 + 0.0050259510\,\epsilon^4 + E(\epsilon)$$

$|E(\epsilon)| < 25 \cdot 10^{-6}$

$E(-1.6) = -0.000017$

$E(0) = 0.000013$

$0 \leq \epsilon \leq 1.6$

$$f_3(\epsilon) = 0.99999443 - 0.33316475\,\epsilon + 0.082501642\,\epsilon^2 - 0.015167643\,\epsilon^3 + 0.0015987224\,\epsilon^4 + E(\epsilon)$$

$|E(\epsilon)| < 15 \cdot 10^{-6}$

$E(0) = -0.000006$

$E(1.6) = 0.000004$

$1.6 \leq \epsilon \leq 2.5$

$$f_3(\epsilon) = 0.62647872 - 0.15945710\,(\epsilon - 1.6) + 0.032887796\,(\epsilon - 1.6)^2 - 0.0045847555\,(\epsilon - 1.6)^3 + E(\epsilon)$$

$|E(\epsilon)| < 10 \cdot 10^{-6}$

$E(1.6) = -0.000003$

$-2.5 \leq \epsilon \leq -1.6$

$$f_4(\epsilon) = 1.5718059 - 0.50042778\,(\epsilon + 1.6) + 0.11221306\,(\epsilon + 1.6)^2 - 0.028770321\,(\epsilon + 1.6)^3 + E(\epsilon)$$

$|E(\epsilon)| < 30 \cdot 10^{-6}$

$E(-1.6) = -0.000019$

$-1.6 \leq \epsilon \leq 0$

$$f_4(\epsilon) = 1.0000037 - 0.24986190\,\epsilon + 0.050698352\,\epsilon^2 - 0.0071338529\,\epsilon^3 + 0.0019878595\,\epsilon^4 + E(\epsilon)$$

$|E(\epsilon)| < 15 \cdot 10^{-6}$

$E(-1.6) = -0.000006$, $E(0) = 0.000004$

$0 \leq \epsilon \leq 1.6$

$$f_4(\epsilon) = 0.99999728 - 0.24992577\,\epsilon + 0.049650271\,\epsilon^2 - 0.0077249373\,\epsilon^3 + 0.00072768106\,\epsilon^4 + E(\epsilon)$$

$|E(\epsilon)| < 10 \cdot 10^{-6}$

$E(0) = -0.000003$, $E(1.6) = 0.000001$

$1.6 \leq \epsilon \leq 2.5$

$$f_4(\epsilon) = 0.70034546 - 0.13843385\,(\epsilon - 1.6) + 0.023139423\,(\epsilon - 1.6)^2 - 0.0027712165\,(\epsilon - 1.6)^3 + E(\epsilon)$$

$|E(\epsilon)| < 10 \cdot 10^{-6}$

$E(1.6) = -0.000001$

$-2.5 \leq \epsilon \leq -1.6$

$$f_5(\epsilon) = 1.4295541 - 0.35599015\,(\epsilon + 1.6) + 0.067011655\,(\epsilon + 1.6)^2 - 0.014116002\,(\epsilon + 1.6)^3 + E(\epsilon)$$

$|E(\epsilon)| < 20 \cdot 10^{-6}$

$E(-1.6) + -0.000008$

$-1.6 \leq \epsilon \leq 0$

$$f_5(\epsilon) = 1.0000008 - 0.19995266\,\epsilon + 0.033598009\,\epsilon^2 - 0.0042806136\,\epsilon^3 + 0.00092920812\,\epsilon^4 + E(\epsilon)$$

$|E(\epsilon)| < 10 \cdot 10^{-6}$

$E(-1.6) = -0.000003$, $E(0) = 0.000001$

$0 \leq \epsilon \leq 1.6$

$$f_5(\epsilon) = 0.99997465 - 0.19955294\,\epsilon + 0.031929205\,\epsilon^2 - 0.0032517350\,\epsilon^3 + E(\epsilon)$$

$|E(\epsilon)| < 35 \cdot 10^{-6}$

$E(0) = -0.000025$
$E(1.6) = -0.000023$

$1.6 \leq \epsilon \leq 2.5$

$$f_5(\epsilon) = 0.74913199 - 0.12192979\,(\epsilon - 1.6) + 0.017239545\,(\epsilon - 1.6)^2 - 0.0018205902\,(\epsilon - 1.6)^3 + E(\epsilon)$$

$|E(\epsilon)| < 10 \cdot 10^{-6}$

$E(1.6) = -0.000001$

$-2.5 \leq \epsilon \leq -1.6$

$$f_6(\epsilon) = 0.13423777 - 0.27260146\,(\epsilon + 1.6) + 0.043983123\,(\epsilon + 1.6)^2 - 0.0078170310\,(\epsilon + 1.6)^3 + E(\epsilon)$$

$|E(\epsilon)| < 15 \cdot 10^{-6}$

$E(-1.6) = -0.000004$

$-1.6 \leq \epsilon \leq 0$

$$f_6(\epsilon) = 0.99996945 - 0.16717386\,\epsilon + 0.022341423\,\epsilon^2 - 0.0043232077\,\epsilon^3 + E(\epsilon)$$

$|E(\epsilon)| < 40 \cdot 10^{-6}$

$E(-1.6) = -0.000032$
$E(0) = -0.000031$

$0 \leq \epsilon \leq 1.6$

$$f_6(\epsilon) = 0.99998524 - 0.16640878\,\epsilon + 0.023003983\,\epsilon^2 - 0.0021178282\,\epsilon^3 + E(\epsilon)$$

$|E(\epsilon)| < 25 \cdot 10^{-6}$

$E(0) = -0.000015$
$E(1.6) = -0.000013$

$1.6 \leq \epsilon \leq 2.5$

$$f_6(\epsilon) = 0.78392780 - 0.10823101\,(\epsilon - 1.6) + 0.011660870\,(\epsilon - 1.6)^2 + E(\epsilon)$$

$|E(\epsilon)| < 40 \cdot 10^{-6}$

$E(1.6) = -0.000032$

$0 \leq \epsilon \leq 1.6$

$$f_8(\epsilon) = 0.99999389 - 0.12489522\,\epsilon + 0.013564977\,\epsilon^2 - 0.0010491184\,\epsilon^3 + E(\epsilon)$$

$|E(\epsilon)| < 15 \cdot 10^{-6}$

$E(1.6) = -0.000005$

$1.6 \leq \epsilon \leq 2.5$

$$f_8(\epsilon) = 0.83057847 - 0.089127204\,(\epsilon - 1.6) + 0.0078138640\,(\epsilon - 1.6)^2 + E(\epsilon)$$

$|E(\epsilon)| < 25 \cdot 10^{-6}$

$E(1.6) = -0.000017$

$0 \leq \epsilon \leq 1.6$

$$f_{10}(\epsilon) = 0.99989680 - 0.099060575\,\epsilon + 0.0075039341\,\epsilon^2 + E(\epsilon)$$

$|E(\epsilon)| < 125 \cdot 10^{-6}$

$E(1.6) = 0.000098$

$1.6 \leq \epsilon \leq 2.5$

$$f_{10}(\epsilon) = 0.85982823 - 0.0706464426\,(\epsilon - 1.6) + E(\epsilon)$$

$|E(\epsilon)| < 750 \cdot 10^{-6}$

$E(1.6) = -0.000684$

APPENDIX C

POLYNOMIAL FITS FOR THE LEADING TERMS IN THE EXPANSIONS FOR FINITE VALUES OF THE RATIO OF VERTICAL TO HORIZONTAL DISTANCE

Near-Zero Expansion (Potential)

$0 \leq A \leq 1$

$$g(A,0) = 1.1544907 + 1.9980015\,A + 0.22411488\,A^2 - 0.25205448\,A^3 + E(A)$$

$$|E(A)| < 60 \cdot 10^{-6}$$

$$\left\{ E(1) = 0.000008 \right\}$$

$1 \leq A \leq 2.1$

$$G(A,0) = 2.0637684 + 3.8292096\,(A-1) - 1.9264769\,(A-1)^2 + 0.21414903\,(A-1)^3 - 0.14931148\,(A-1)^4 + 0.056566782\,(A-1)^5 + E(A)$$

$$|E(A)| < 50 \cdot 10^{-6}$$

$$\left\{ \begin{array}{l} E(1) = 0.000014 \\ E(2.1) = 0.000047 \end{array} \right\}$$

$2.1 \leq A \leq 3$

$$G(A,0) = 4.1023310 - 0.014383310\,(A-2.1) - 1.5755990\,(A-2.1)^2 + 0.18698199\,(A-2.1)^3 + 0.058979955\,(A-2.1)^4 + E(A)$$

$$|E(A)| < 20 \cdot 10^{-6}$$

$$\left\{ \begin{array}{l} E(2.1) = -0.000011 \\ E(3) = 0.000012 \end{array} \right\}$$

$3 \leq A \leq 5$

$$G(A,0) = 2.9881196 - 2.2236879\,(A-3) - 0.79598019\,(A-3)^2 + 0.39190768\,(A-3)^3 + 0.023937640\,(A-3)^4 - 0.013320342\,(A-3)^5 + E(A)$$

$$|E(A)| < 65 \cdot 10^{-6}$$

$$\left\{ E(3) = -0.000025 \right\}$$

Near-Zero Expansion (Derivative)

$0 \leq A \leq 1$

$$h(A,0) = 2.0000031 + 0.42257532\,A - 0.66467931\,A^2 - 0.12212829\,A^3 + 0.054568171\,A^4 + E(A)$$

$$|E(A)| < 10 \cdot 10^{-6}$$

$$\{E(1) = -0.000006\}$$

$1 \leq A \leq 2.1$

$$H(A,0) = 3.8307581 - 3.8920436\,(A-1) + 0.90918151\,(A-1)^2 - 1.3445198\,(A-1)^3 + 1.2526896\,(A-1)^4 - 0.56621765\,(A-1)^5 + 0.11452397\,(A-1)^6 + E(A)$$

$$|E(A)| < 40 \cdot 10^{-6}$$

$$\begin{cases} E(1) = -0.000022 \\ E(2.1) = +0.000015 \end{cases}$$

$2.1 \leq A \leq 3$

$$H(A,0) = -0.014925130 - 3.1415572\,(A-2.1) + 0.51558135\,(A-2.1)^2 + 0.31359777\,(A-2.1)^3 - 0.043664524\,(A-2.1)^4 + E(A)$$

$$|E(A)| < 30 \cdot 10^{-6}$$

$$\begin{cases} E(2.1) = -0.000024 \\ E(3) = 0.000022 \end{cases}$$

$3 \leq A \leq 5$

$$H(A,0) = -2.2247429 - 1.5803259\,(A-3) + 1.1424351\,(A-3)^2 + 0.12911733\,(A-3)^3 - 0.073811662\,(A-3)^4 - 0.0056757478\,(A-3)^5 + 0.0022340944\,(A-3)^6 + E(A)$$

$$|E(A)| < 30 \cdot 10^{-6}$$

$$\{E(3) = 0.000020\}$$

Near-One Expansion (Potential)

$0 \leq A \leq 1$

$$g(A,1) = 2.9171383 - 0.086882334\,A - 0.90711445\,A^2 + 0.33854153\,A^3 - 0.031389738\,A^4 + E(A)$$

$|E(A)| < 45 \cdot 10^{-6}$

$E(1) = 0.000028$

$1 \leq A \leq 1.8$

$$G(A,1) = 1.8400312 + 0.16427911\,(A-1) - 1.5778025\,(A-1)^2 + 1.2836527\,(A-1)^3 - 0.56383902\,(A-1)^4 + 0.12545148\,(A-1)^5 + E(A)$$

$|E(A)| < 20 \cdot 10^{-6}$

$E(1) = 0.000009$

$E(1.8) = 0.000001$

$1.8 \leq A \leq 3$

$$G(A,1) = 1.4290456 - 0.79388355\,(A-1.8) + 0.023527874\,(A-1.8)^2 + 0.23458663\,(A-1.8)^3 - 0.11577594\,(A-1.8)^4 + 0.021430690\,(A-1.8)^5 + E(A)$$

$|E(A)| < 20 \cdot 10^{-6}$

$E(1.8) = -0.000004$

$E(3) = 0.000016$

$3 \leq A \leq 5$

$$G(A,1) = 0.66112296 - 0.41639898\,(A-3) + 0.18942493\,(A-3)^2 - 0.040367529\,(A-3)^3 + 0.00073725892\,(A-3)^4 + 0.00074203049\,(A-3)^5 + E(A)$$

$|E(A)| < 30 \cdot 10^{-6}$

$E(3) = 0.000014$

Near-One Expansion (Derivative)

$0 \leq A \leq 1$

$$h(A,1) = 2.8284779 - 1.8750697\,A + 0.0045364752\,A^2 + 0.36642281\,A^3 - 0.10645884\,A^4 + E(A)$$

$|E(A)| < 60 \cdot 10^{-6}$

$E(1) = -0.000053$

$1 \leq A \leq 1.8$

$$H(A,1) = 2.0053627 - 3.0194857\,(A-1) + 2.4812103\,(A-1)^2 - 1.5964116\,(A-1)^3 + 0.88156561\,(A-1)^4 - 0.26310315\,(A-1)^5 + E(A)$$

$|E(A)| < 40 \cdot 10^{-6}$

$E(1) = -0.000021$

$E(1.8) = -0.000021$

$1.8 \leq A \leq 3$

$$H(A,1) = 0.63527320 - 0.84547871\,(A-1.8) + 0.71240566\,(A-1.8)^2 - 0.30982833\,(A-1.8)^3 + 0.075579468\,(A-1.8)^4 - 0.0090482946\,(A-1.8)^5 + E(A)$$

$|E(A)| < 20 \cdot 10^{-6}$

$E(1.8) = -0.000009$

$E(3) = -0.000001$

$3 \leq A \leq 5$

$$H(A,1) = 0.24537160 - 0.044912902\,(A-3) + 0.091506445\,(A-3)^2 - 0.066964068\,(A-3)^3 + 0.020788426\,(A-3)^4 - 0.0025195079\,(A-3)^5 + E(A)$$

$|E(A)| < 25 \cdot 10^{-6}$

$E(3) = -0.000016$

Near-Seven-Quarters Expansion (Potential)

$0 \leq A \leq 1$

$$g(A,7/4) = 3.8062113 - 2.6292506\,A + 0.08310758\,A^2 + 0.71818628\,A^3 - 0.39167057\,A^4 + 0.078722785\,A^5 + E(A)$$

$|E(A)| \leq 20 \cdot 10^{-6}$

$E(1) = 0.000020$

$1 \leq A \leq 2$

$$G(A,7/4) = 1.4809815 - 0.79116564\,(A-1) - 0.47568652\,(A-1)^2 + 0.97274634\,(A-1)^3 - 0.61746058\,(A-1)^4 + 0.15520651\,(A-1)^5 + E(A)$$

$|E(A)| < 45 \cdot 10^{-6}$

$E(1) = 0.000032$

$E(2) = 0.000040$

$2 \leq A \leq 3$

$$G(A,7/4) = 0.72459428 - 0.52154760\,(A-2) + 0.26664779\,(A-2)^2 - 0.076093178\,(A-2)^3 + 0.0081860973\,(A-2)^4 + E(A)$$

$|E(A)| < 15 \cdot 10^{-6}$

$E(2) = 0.000013$

$E(3) = 0.000010$

$3 \leq A \leq 5$

$$G(A,7/4) = 0.40171333 - 0.18141188\,(A-3) + 0.082011657\,(A-3)^2 - 0.026231297\,(A-3)^3 + 0.0038098253\,(A-3)^4 + E(A)$$

$|E(A)| < 85 \cdot 10^{-6}$

$E(3) = 0.000084$

Near-Seven-Quarters Expansion (Derivative)

$0 \leq A \leq 1$

$$h(A,7/4) = 4.03112830 - 4.43098650\,A + 2.30502990\,A^2 - 0.37921324\,A^3 - 0.16673206\,A^4 + 0.072409875\,A^5 + E(A)$$

$|E(A)| < 30 \cdot 10^{-6}$

$E(1) = 0.000028$

$1 \leq A \leq 2$

$$H(A,7/4) = 1.80347950 - 2.40290510\,(A-1) + 2.50943380\,(A-1)^2 - 1.85077490\,(A-1)^3 + 0.89663576\,(A-1)^4 - 0.20878570\,(A-1)^5 + E(A)$$

$|E(A)| \leq 90 \cdot 10^{-6}$

$E(1) = 0.000081$

$E(2) = 0.000049$

$2 \leq A \leq 3$

$$H(A,7/4) = 0.74709173 - 0.38813724\,(A-2) + 0.27309180\,(A-2)^2 - 0.15160002\,(A-2)^3 + 0.039336945\,(A-2)^4 + E(A)$$

$|E(A)| < 45 \cdot 10^{-6}$

$E(2) = 0.000041$

$E(3) = 0.000034$

$3 \leq A \leq 5$

$$H(A,7/4) = 0.51968616 - 0.13928528\,(A-3) + 0.036160438\,(A-3)^2 - 0.0092761385\,(A-3)^3 + 0.0013868976\,(A-3)^4 + E(A)$$

$|E(A)| < 65 \cdot 10^{-6}$

$E(3) = 0.000063$

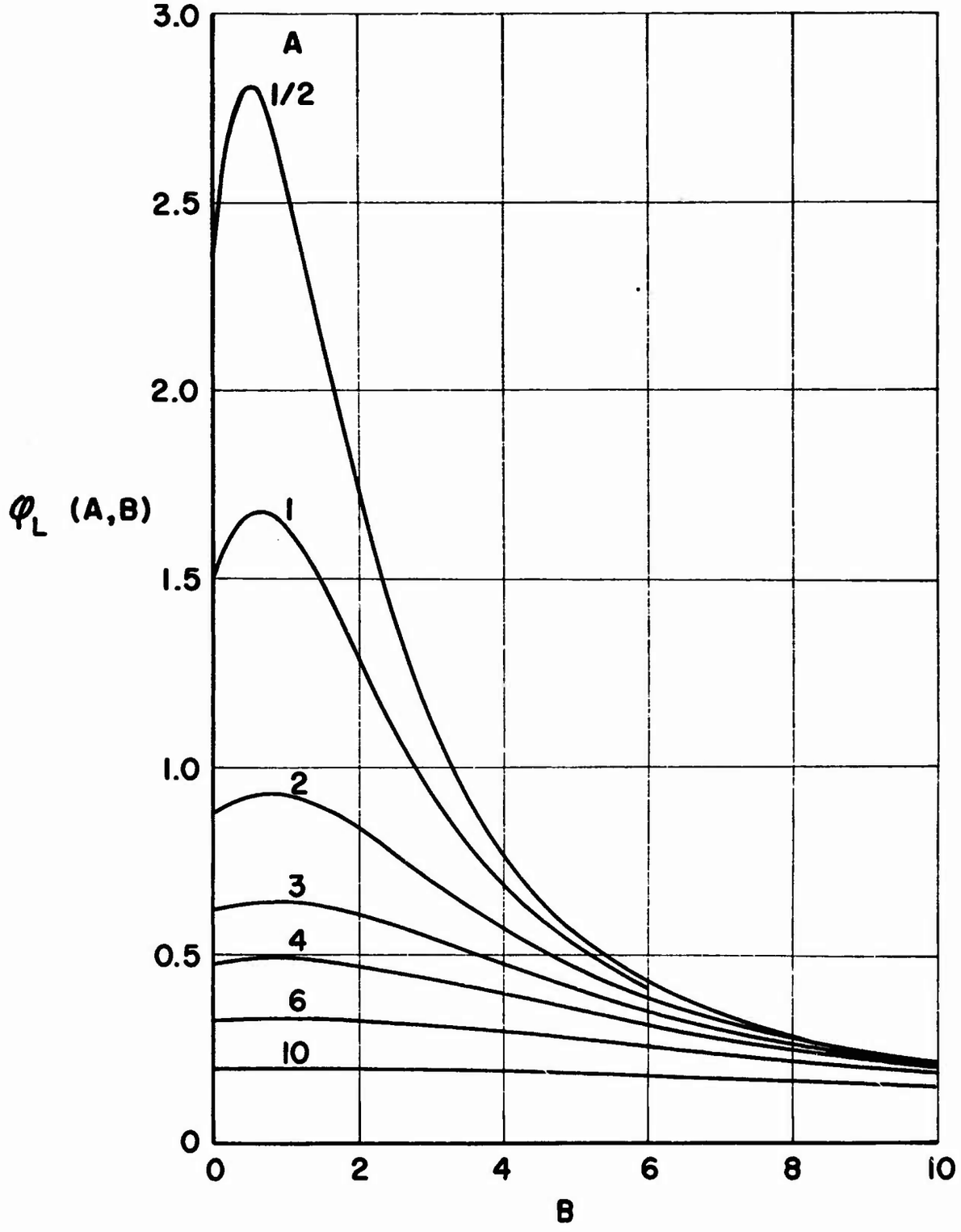

Figure 7. - The potential φ_L.

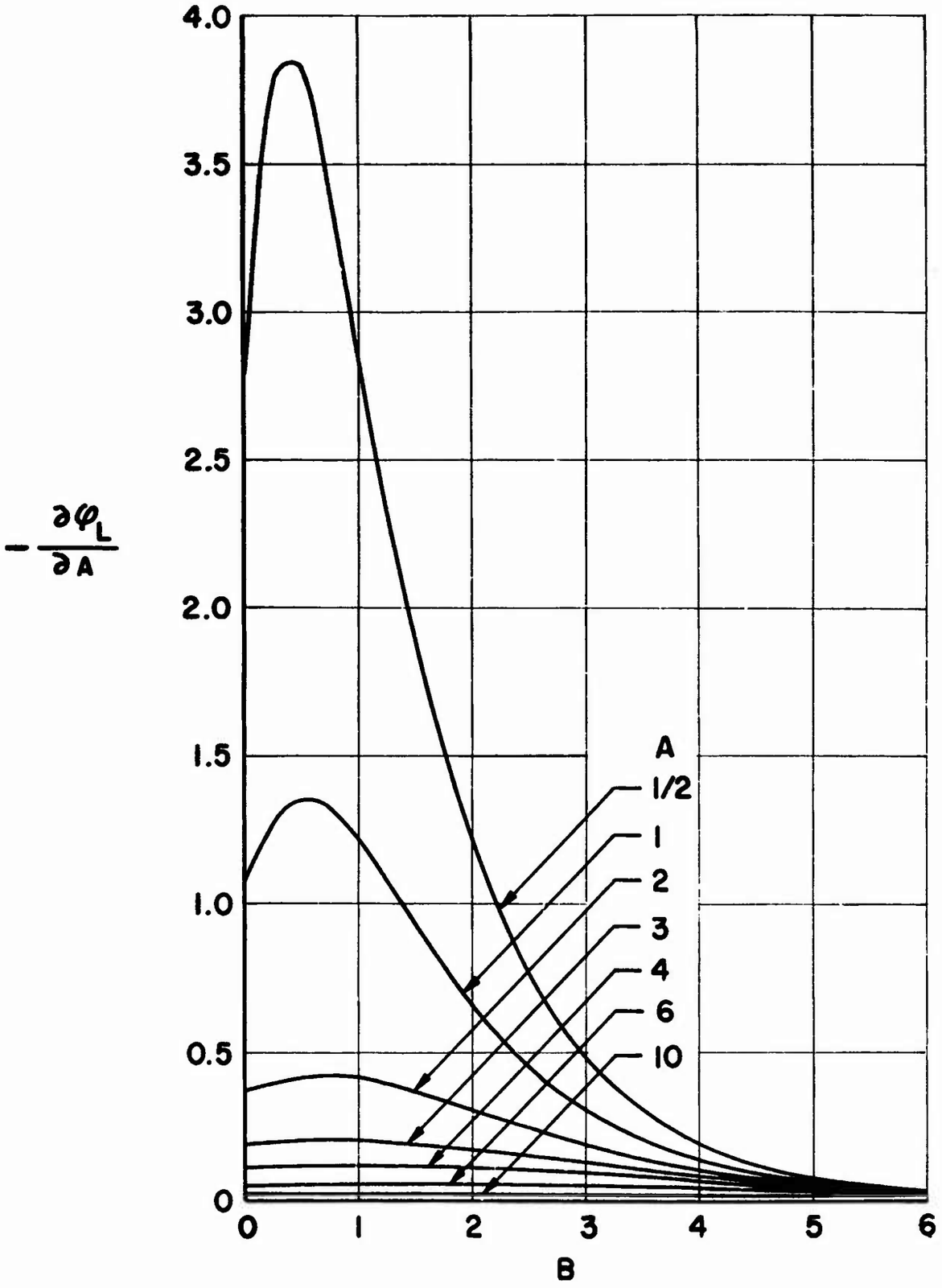

Figure 8. - The A-derivative of φ_L.

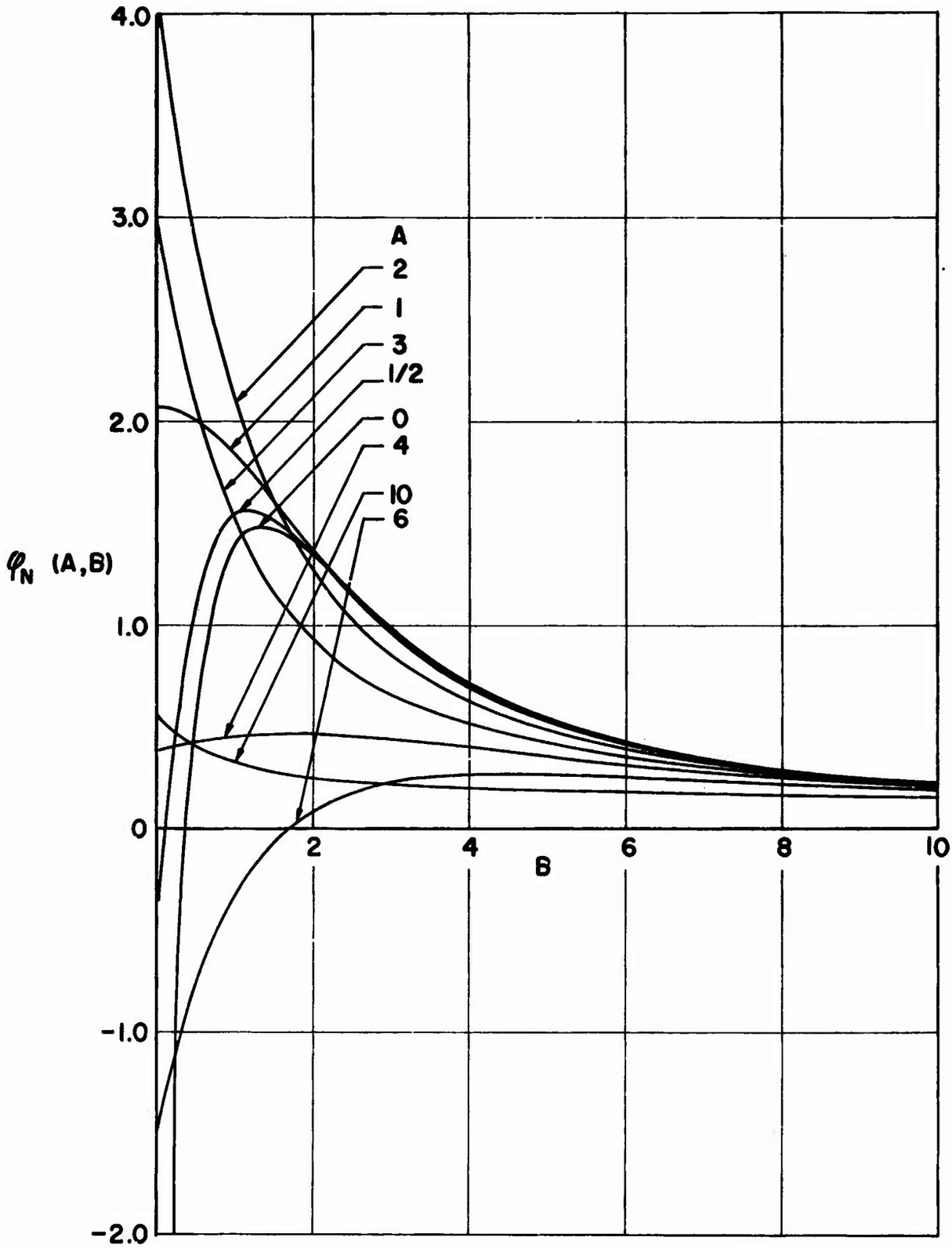

Figure 9. - The potential φ_N.

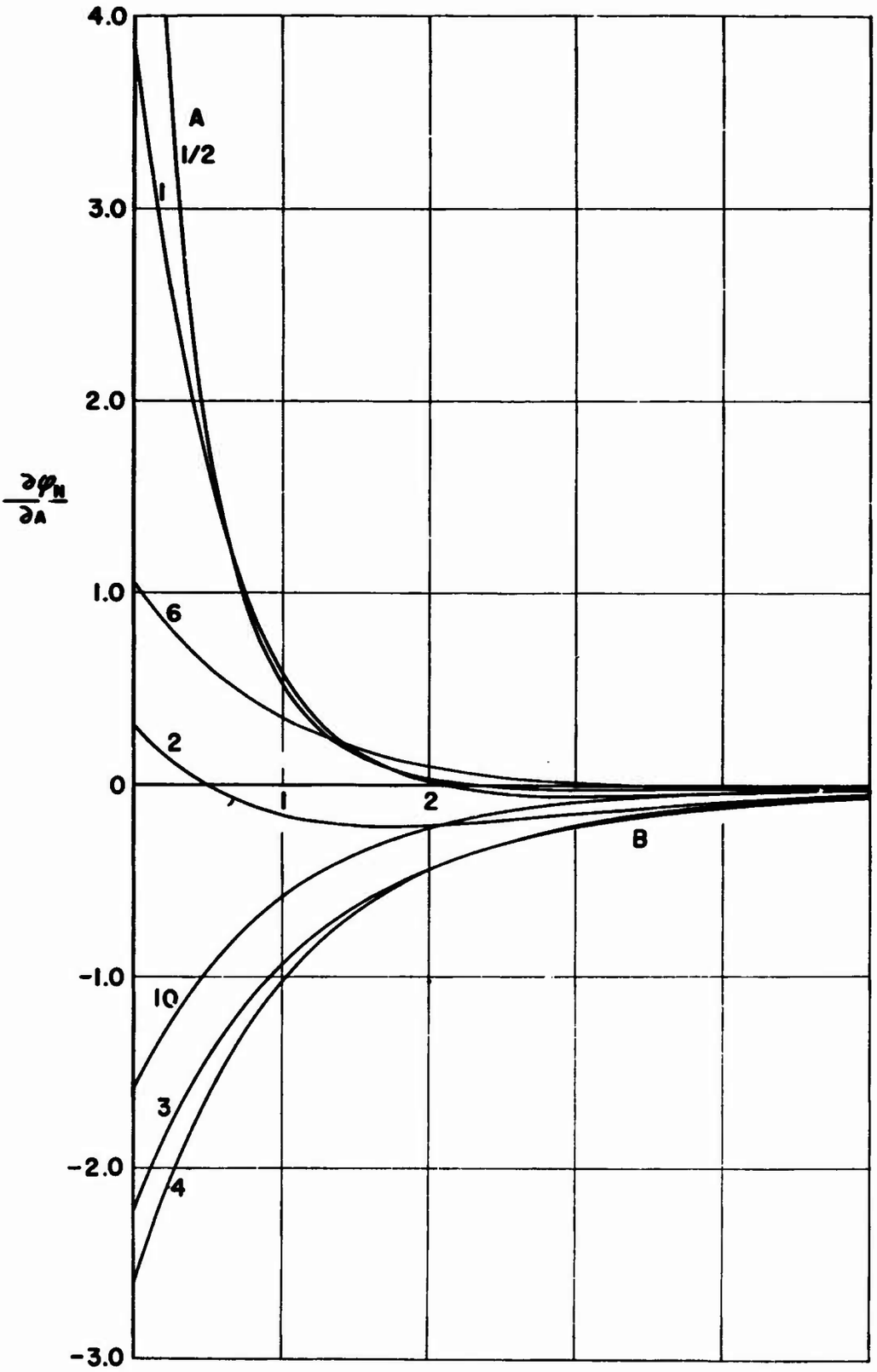

Figure 10. - The A-derivative of φ_N.

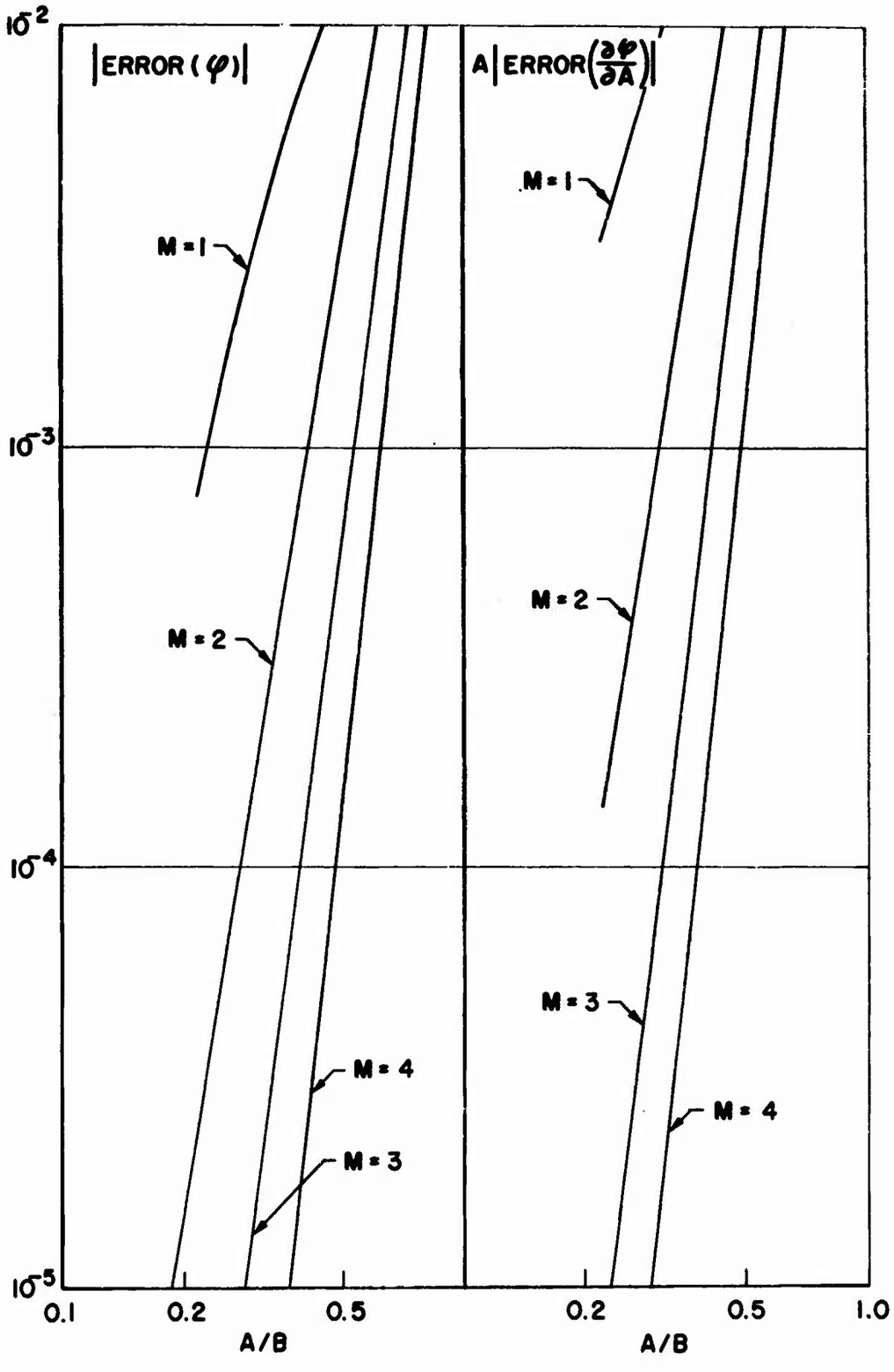

Figure 11. - Maximum errors in the potential φ_N and its A-derivative calculated by the exponential-integral expansion.

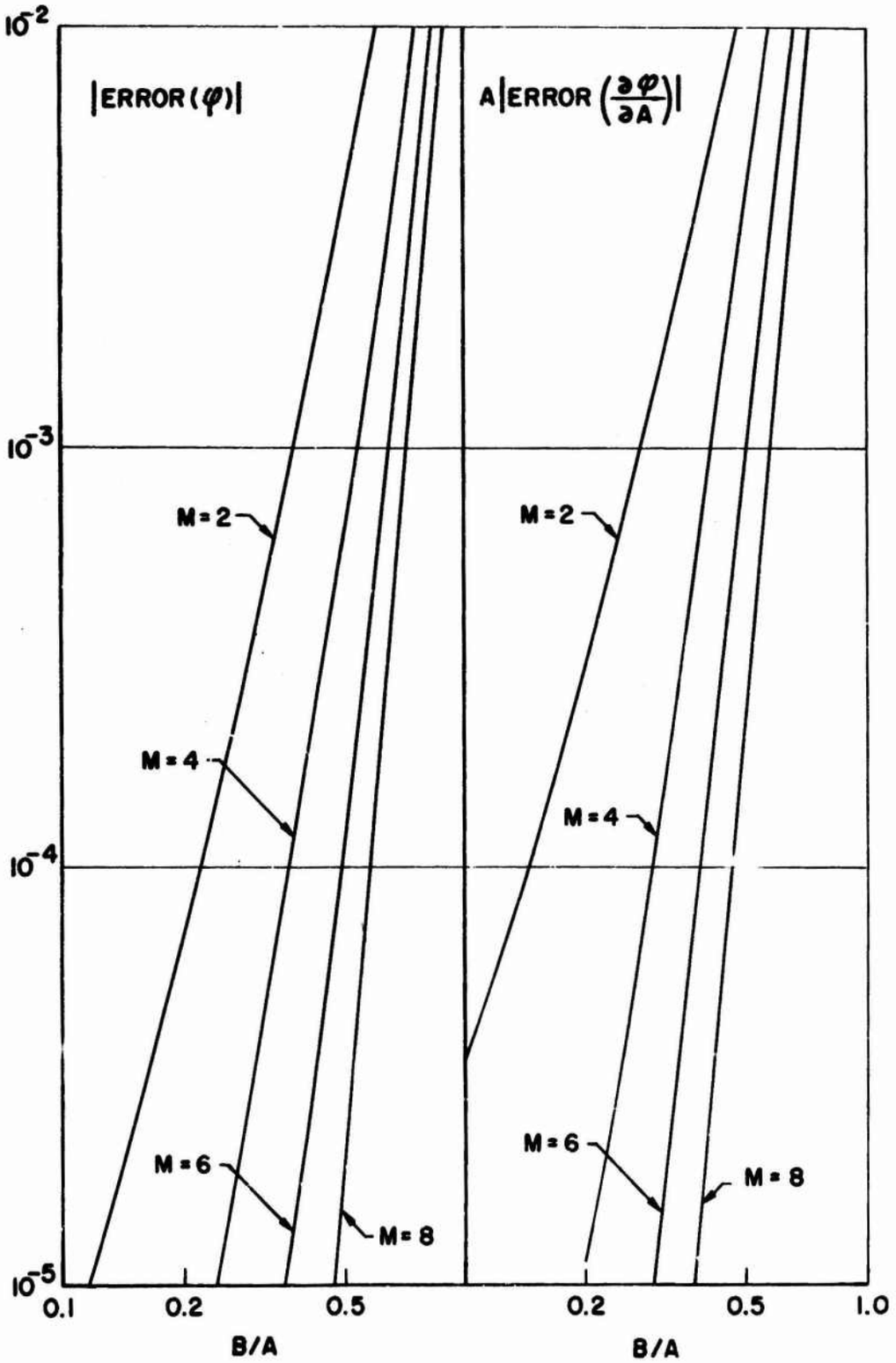

Figure 12. - Maximum errors in the potential φ_N and its A-derivative calculated by the near-zero expansion.

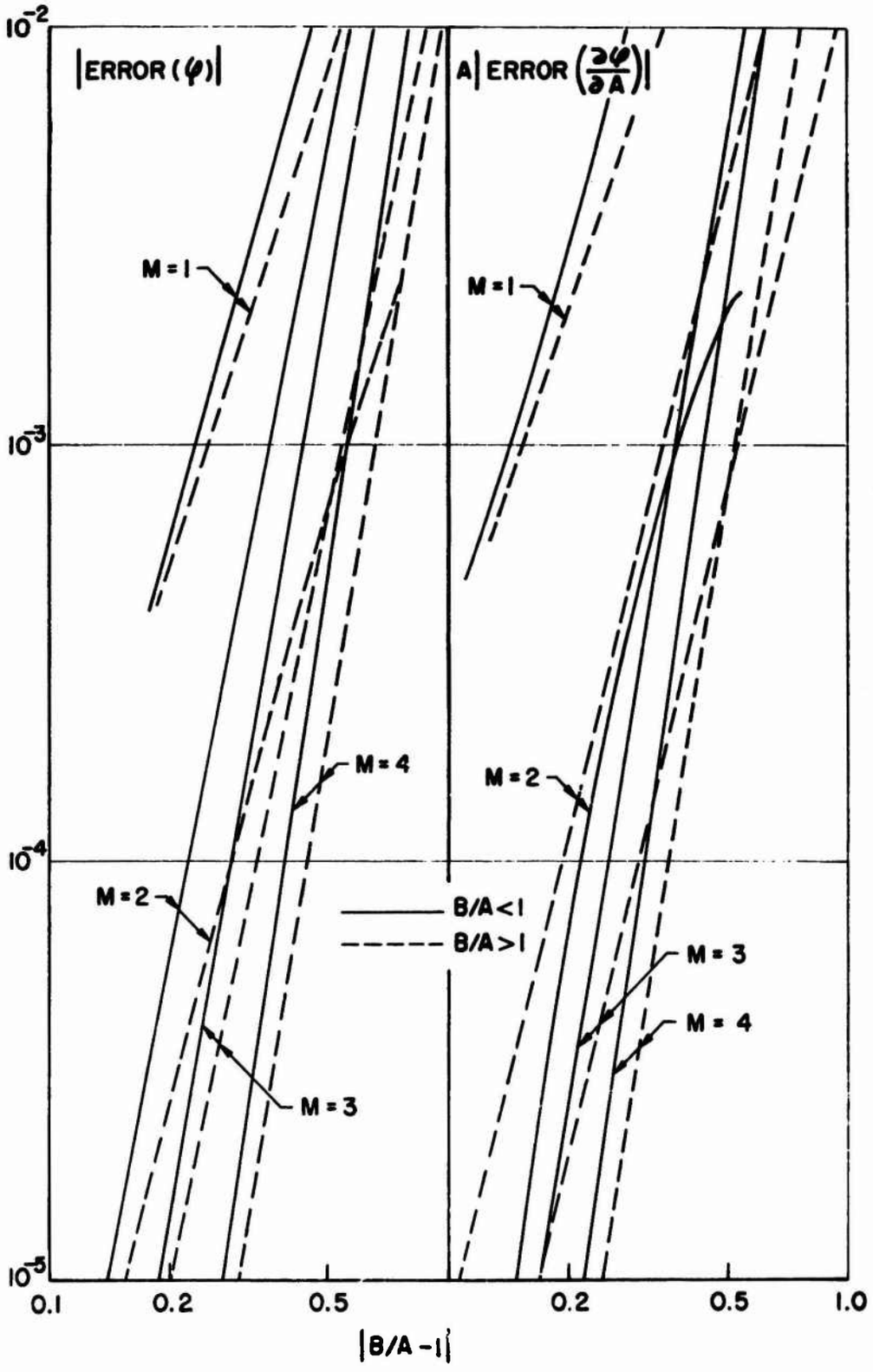

Figure 13. Maximum errors in the potential φ_N and its A-derivative calculated by the near-one expansion.

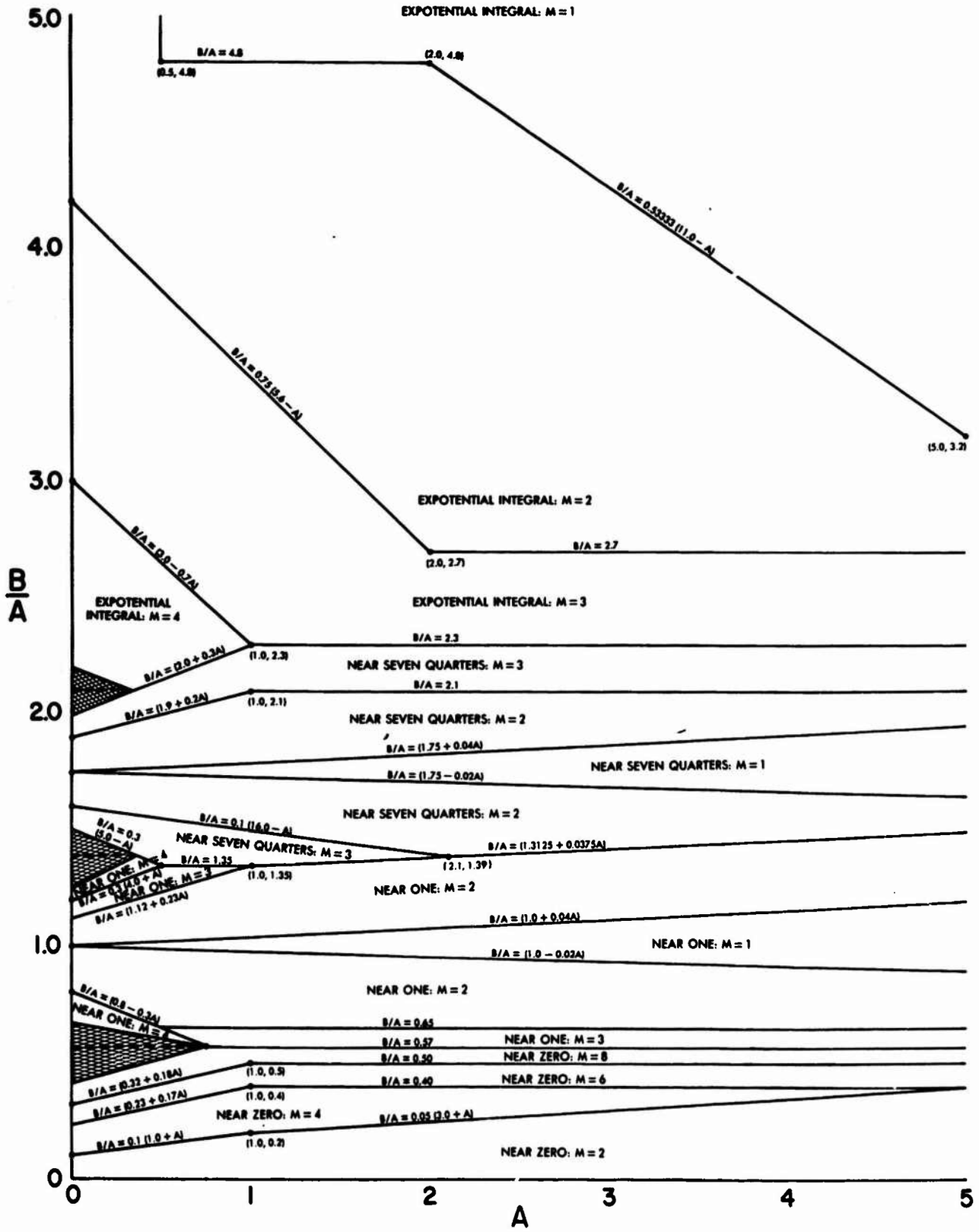

Figure 14. - Ranges of values of A and B/A where certain numbers of terms of the several expansions are to be used to obtain an accuracy of 0.001 in computing the potential φ_N and its A-derivative.

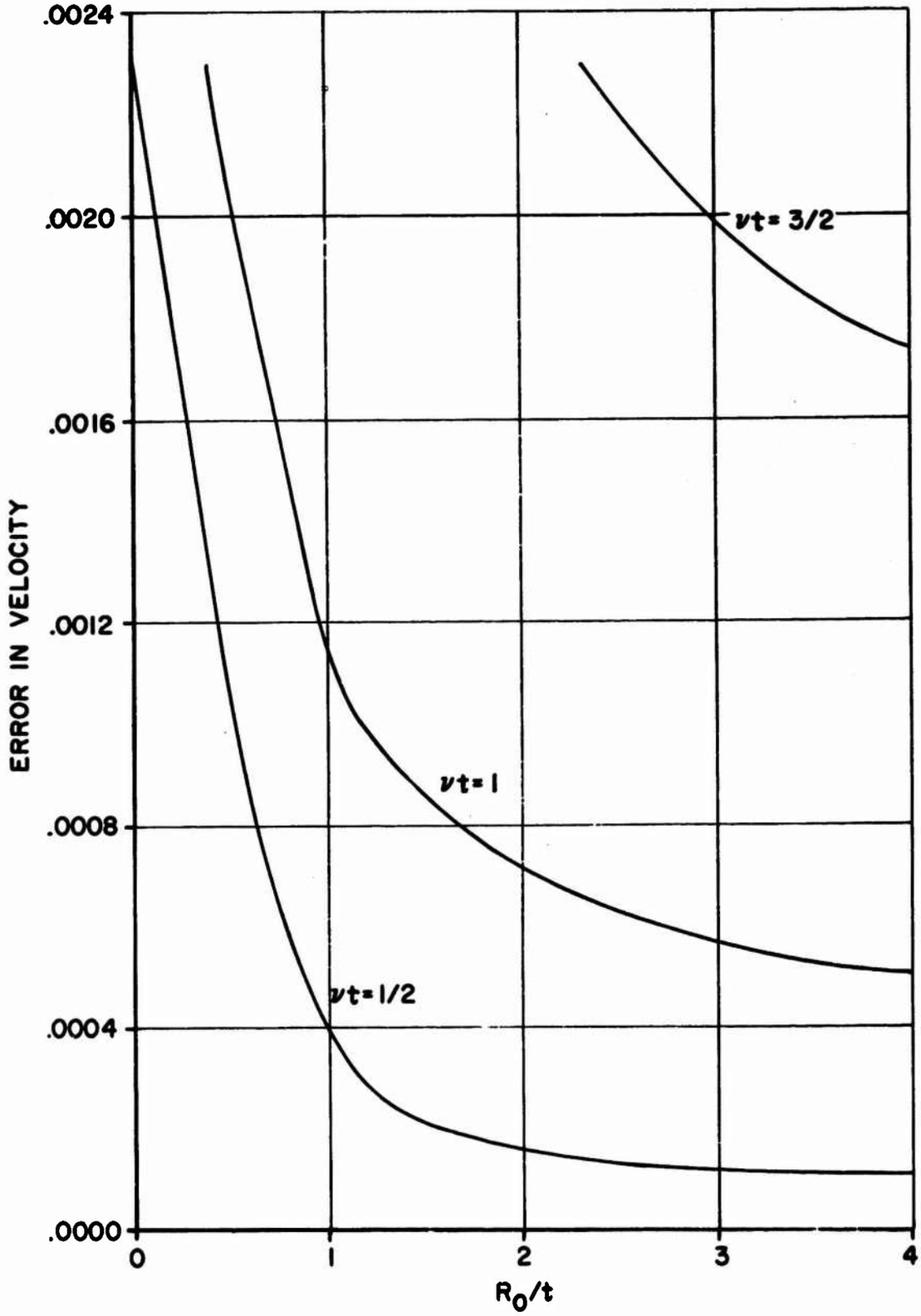

Figure 15. - Maximum errors in velocity calculated by the second-order multipole expansion. (a) $h_0/t = 1$.

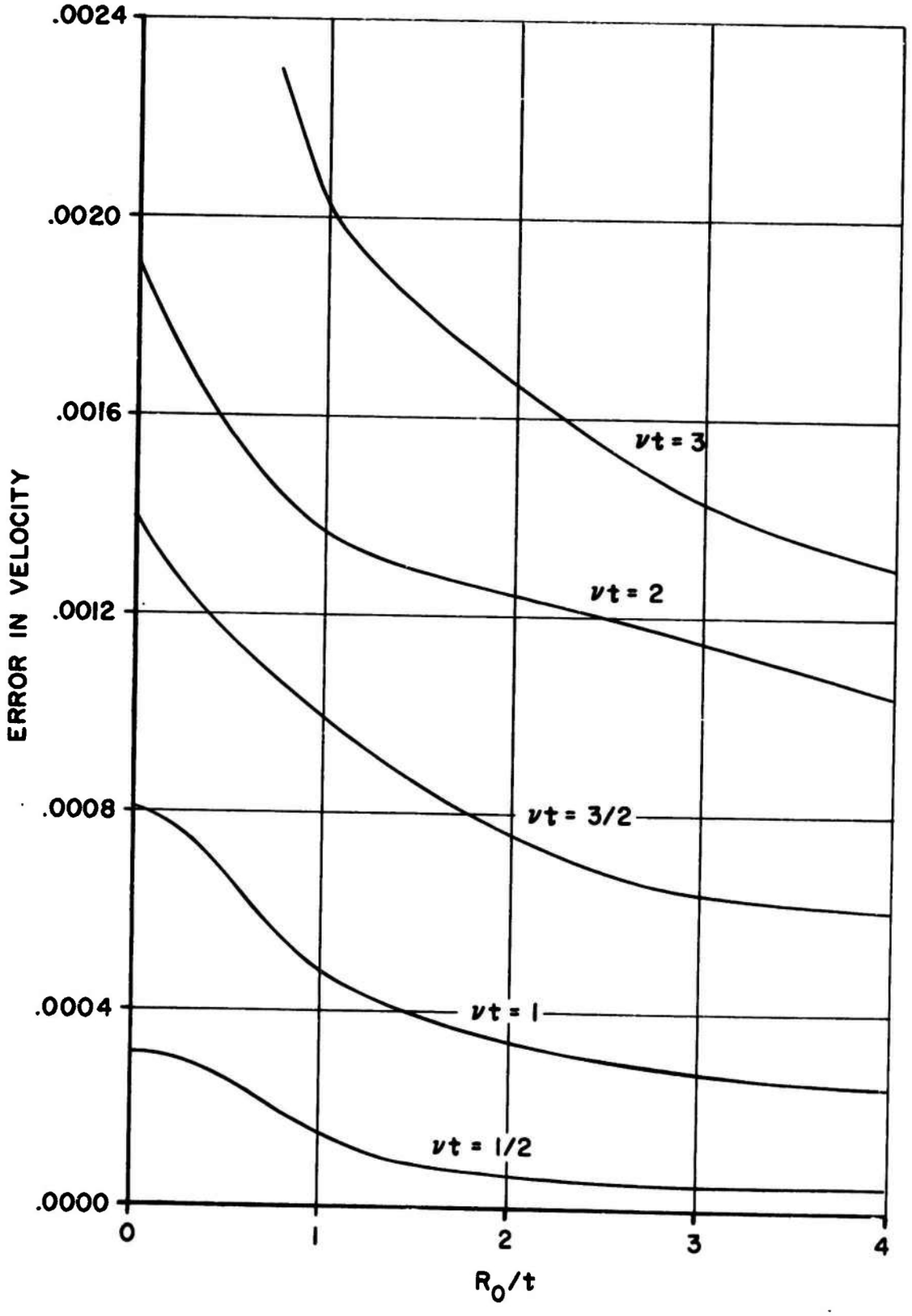

Figure 15. - Continued. (b) $h_o/t = 3/2$.

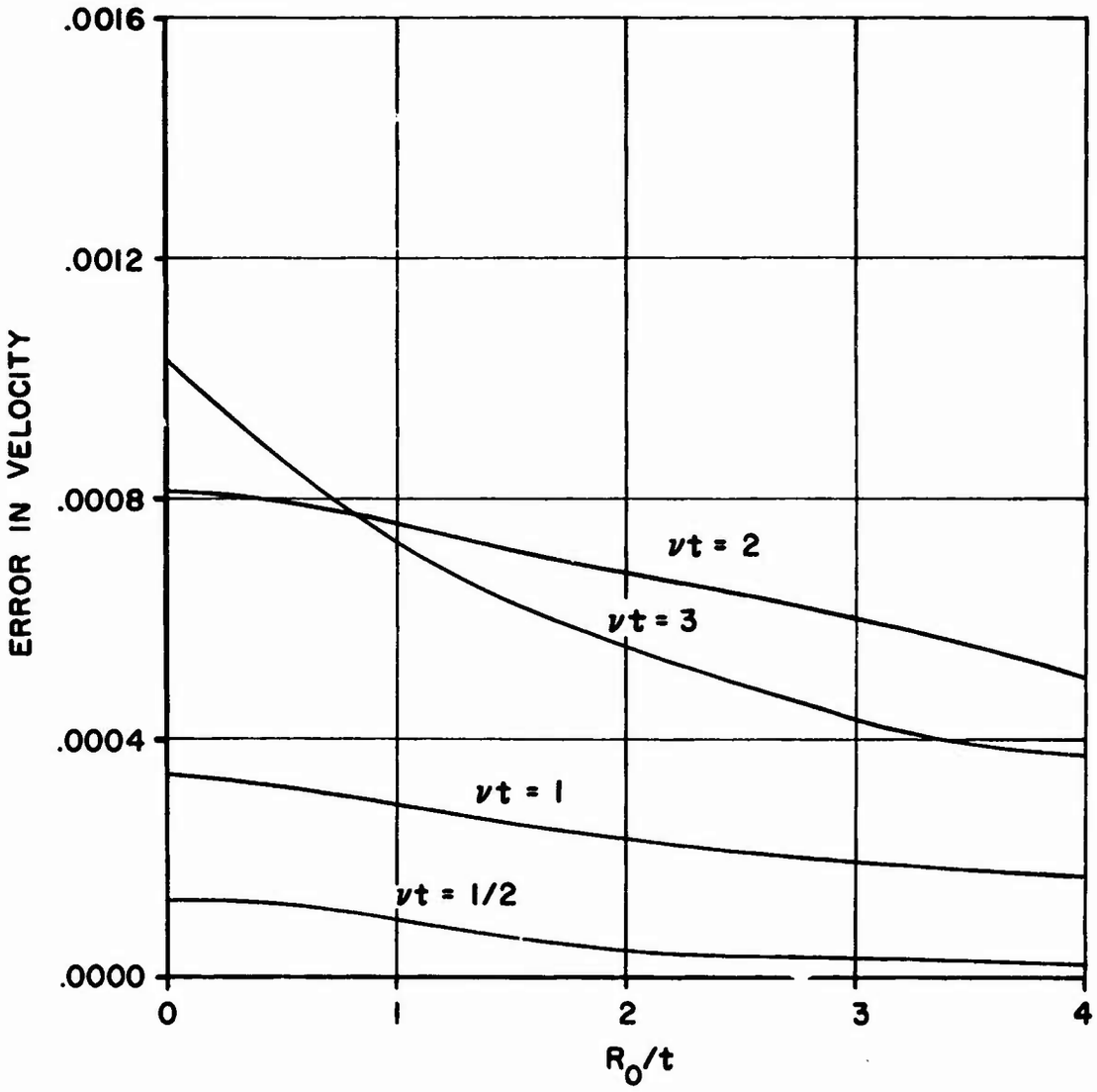

Figure 15. - Continued. (c) $h_o/t = 2$.

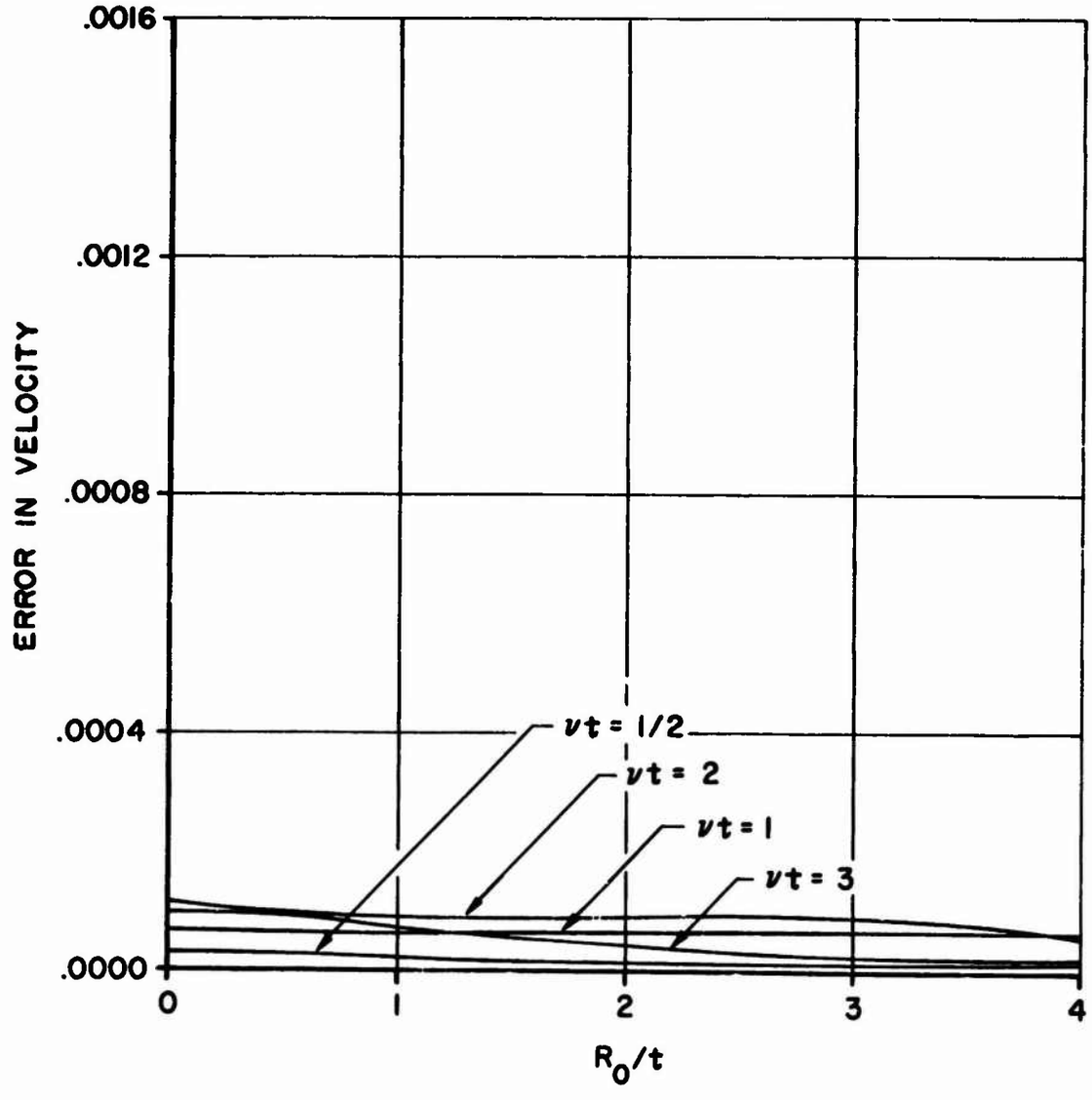

Figure 15. - Continued. (d) $h_0/t = 3$.

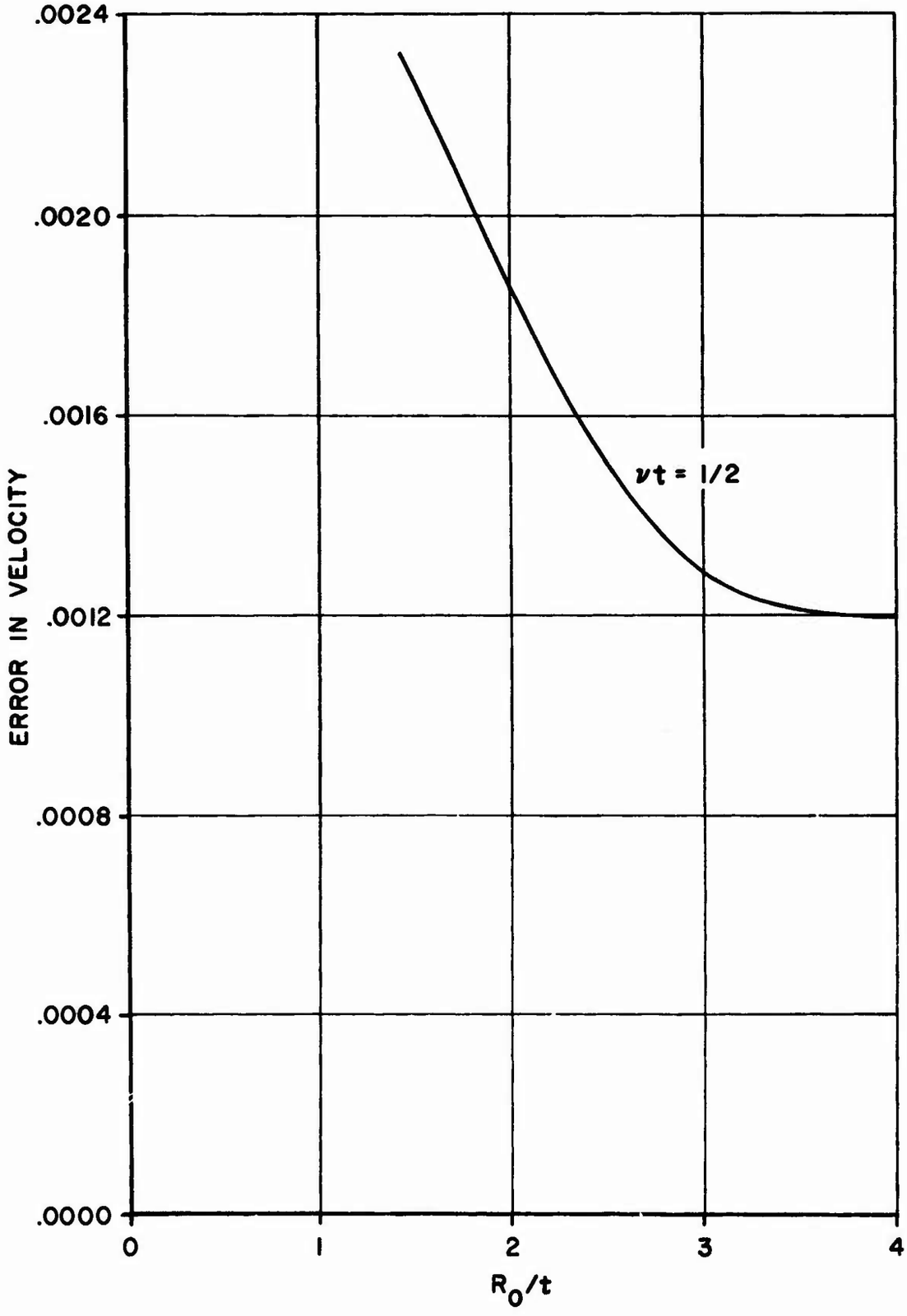

Figure 16. - Maximum errors in velocity calculated by the zeroth-order multipole expansion. (a) $h_0/t = 2$.

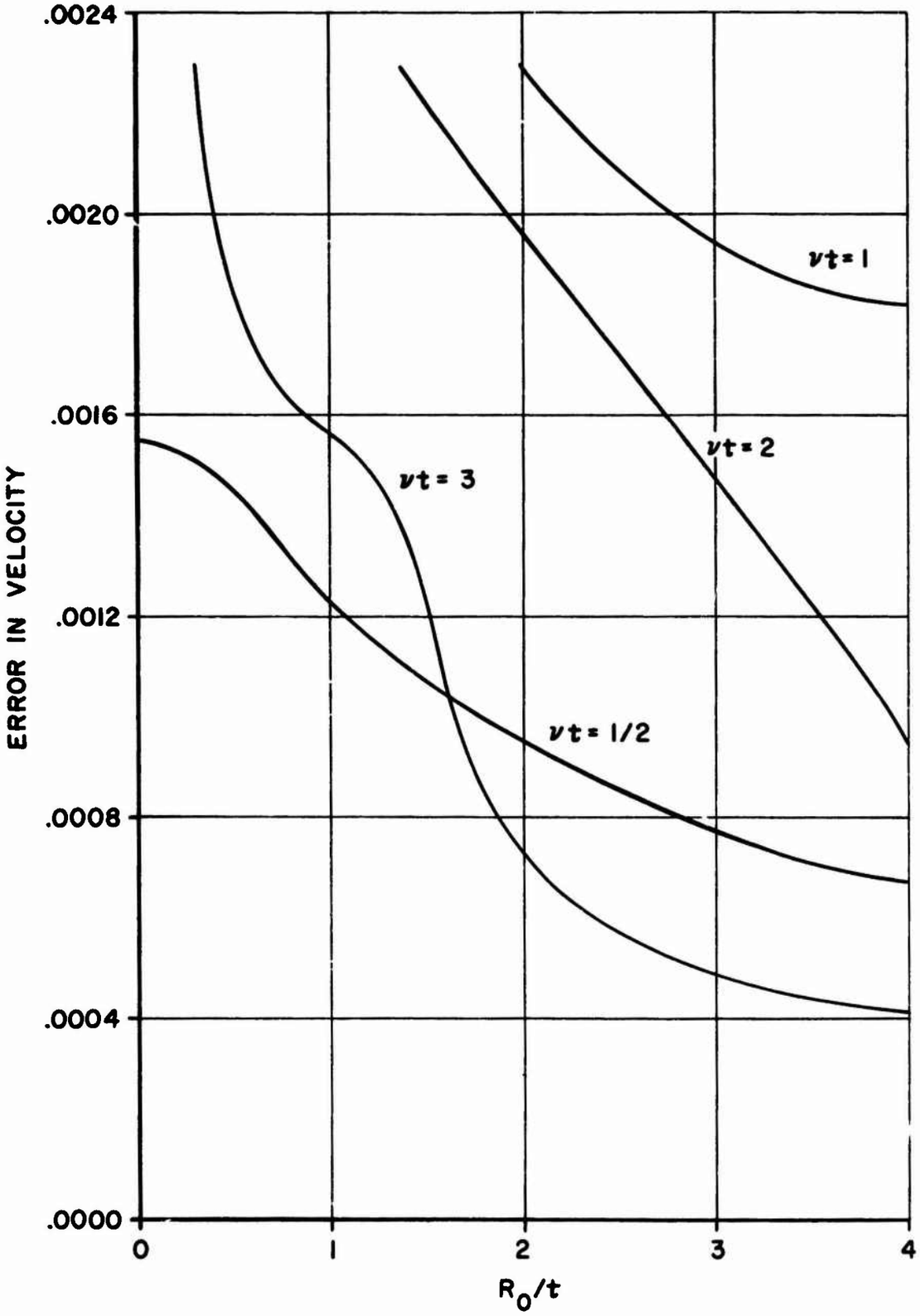

Figure 16. - Continued. (b) $h_o/t = 3$.

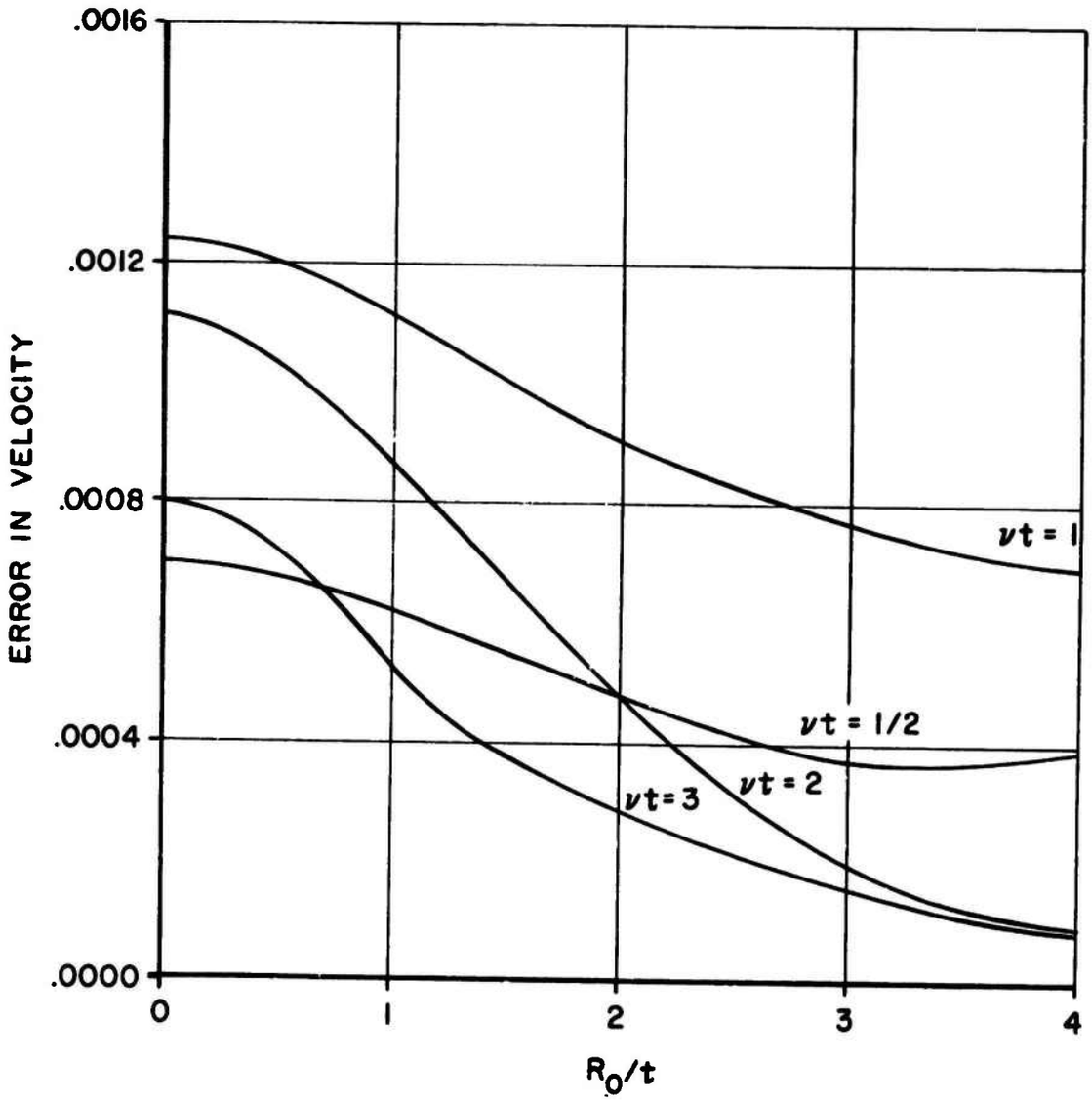

Figure 16. - Continued. (c) $h_0/t = 4$.

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4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

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5. AUTHOR(S) (Last name, first name, initial)

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13. ABSTRACT

This report presents formulas for the rapid evaluation of the potential and potential gradient due to an oscillating point source near a free surface. Also presented are expressions that represent the integrals of these formulas over a general plane quadrilateral. These expressions could form the basis of a computer program for solving the title problem, and the general features of such a method of solution are outlined.

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
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